APPLIED METHODS OF STATISTICAL ANALYSIS.

STATISTICAL COMPUTATION AND SIMULATION

PROCEEDINGS
OF THE INTERNATIONAL WORKSHOP

18-20 September 2019

Novosibirsk

2019

Estimation of the present values of net premiums and life annuities for the different actuarial models

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Abstract

The paper deals with the estimation problem of the actuarial present values of the continuous individual net premiums and connected with these characteristics life annuities. We considered the following actuarial models: the whole life insurance, n-year term life insurance, q-year deferred life insurance, and n-year endowment life insurance. We synthesize nonparametric estimators of net premiums and life annuities for these statuses. The main parts of the asymptotic mean square errors of the estimators and their limit distributions are found. The simulations show that the empirical mean square errors of estimators decrease when the sample size increases. Also, when the model distribution is changed, the nonparametric estimators are more adaptable in comparison with parametric estimators, oriented on the best results only for the given distributions.

Keywords: nonparametric estimation; life insurance; net premium; life annuity; asymptotic normality; bias; mean squared error.

Introduction

One of the main issues addressed in actuarial mathematics is to find the "right" ratio between premiums and benefits, aided calculation of net premiums intended to cover damages and giving zero average income of the insurance company. Section devoted to this area in the monograph "Actuarial Mathematics" [8], in which the calculation of net premiums was based on the use of mortality tables. Interesting results based on this approach have been prepared in papers [5, 10, 13, 16, 34, 39, 42]. Modern development of theory of insurance is strongly required the use of complex mathematical models phenomena and processes taking place in this area. Note the results obtained in this direction in the papers [1, 4, 6, 18, 19, 22, 37]. Alternative solution is to build estimators of net premium functionals on the base of information containing in a sample of individuals' lifetimes. Here we develop this idea embedded in the articles [20], [25]-[33]. The second part of the paper deals with the study of life annuities estimators.

1 Individual Whole Life Insurance: Net Premiums and Life Annuities

In long-term insurance the calculations of tariff rates take into account change of money value because the sum of S dollars after t years turns to the sum $S\,e^{\delta t}$ dollars, where δ is instantaneous interest rate. The whole life insurance is example of long-term insurance; in this situation the person pays p dollars to the insurance company, and the company pays b dollars to successors of the insured after his death. Though the premium p is less, than b, the company will receive the necessary sum b, since the premium is paid at the moment of the conclusion of the contract, and the payment is done great later. We will use designations of actuarial mathematics later on. Let random variable X denote the future lifetime, x be the age of the insured at the moment of policy issue, $T(x) = X - x$ denote the residual time of life. In time $T(x)$ premium, p, will turn in the sum, $p\,e^{\delta T(x)}$, and in this case the income of the company will be equal to

$$p\,e^{\delta T(x)} - b.$$

To have the required sum b dollars at the moment of client death, the insurance company must receive $b\,e^{-\delta T(x)}$ dollars at the time of policy issue. In economic terms, the sum $b\,e^{-\delta T(x)}$ expresses discounted value of the future insurance payment. As the above mentioned this sum is a random variable, so it is natural to take as net premium its average the symbol of the expectation. In actuarial science the benefit b is accepted as a unit payment, that is, $b = 1$, and the net premium of the whole life insurance $\overline{A}_x$ is equal to $\mathbf{E}\{e^{-\delta T(x)}\}$:

$$\overline{A}_x = \mathbf{E}\{e^{-\delta T(x)}\} = -\int_0^\infty e^{-\delta t}\, d\mathbf{P}\{T(x) > t | T(x) > 0\} =$$

$$= \frac{-\int_0^\infty e^{-\delta t}\, d\mathbf{P}\{T(x) > t \cap T(x) > 0\}}{\mathbf{P}\{T(x) > 0\}} =$$

$$= \frac{\int_0^\infty e^{-\delta t}\, I_{T(x)}(t > 0) dF_{T(x)}(t)}{S_{T(x)}(0)} = \frac{\Phi(x, \delta)}{S_{T(x)}(0)}, \tag{1}$$

where $F_{T(x)}(t) = \mathbf{P}(T(x) \leq t)$ is the distribution function of the random variable $T(x)$, $S_{T(x)}(t) = 1 - F(t) = \mathbf{P}(T(x) > t)$ is the survival function, $I_{T(x)}(t > 0) = I(T(x) > 0)$, $I(A)$ is the indicator of set A.

It is known that life annuities are closely related to the corresponding net premiums (see [4]). The idea of life annuity in accordance with ([4], p. 170) is this: from the moment t = 0 an individual once a year begins to get a certain money, which we take as the unit of money, and payments are made only for the lifetime of an individual. As the calculation of the characteristics of life annuity is based on the characteristics of the respective type of insurance, the average total cost of the present continuous

annuity is defined by the following formula (see [4], p. 184):

$$\overline{a}_x(\delta) = \frac{1 - \overline{A}_x}{\delta}, \tag{2}$$

where $\overline{A}_x$ is a net premium (the average of the present value of a single sum of money in the insurance lifetime at the age x. Let us introduce the random variable

$$z(x) = \frac{1 - e^{-\delta T(x)}}{\delta}, \quad T(x) > 0. \tag{3}$$

Then, by averaging the random variable $z(x)$ (3), we get the formula of the whole life annuity (see [8, 21, 30]):

$$\overline{a}_x(\delta) = \mathbf{E}z(x) = \frac{1}{\delta}\left(1 - \frac{\Phi(x,\delta)}{S_{T(x)}(0)}\right). \tag{4}$$

2 Collective Life Insurance

A useful abstraction in the collective life insurance is that of "*status* for which there are definitions of survival and failure" [8]. Consider m members of ages $(x_1, \ldots, x_m)$ who desire to buy an insurance policy. Denote the future lifetime of the k-th individual by $T(x_k) = X_k - x_k$. Let us put in a correspondence a status U with its future lifetime $T(U)$ and with a set of numbers $T(x_1), \ldots, T(x_m)$ [20].

In the papers [25]-[33] were considered cases of a joint-life status and a last-survivor status.

The joint-life status is denoted by $U := x_1 : \ldots : x_m$ and is considered as failed upon the first death, i.e.,

$$T(U) = \min\left(T(x_1), \ldots, T(x_m)\right).$$

It is evident that the probability

$$\mathbf{P}\{T(U) > t\} = \mathbf{P}\{\min(T(x_1), \ldots, T(x_m)) > t\} = \mathbf{P}\{T(x_1) > t, \ldots, T(x_m) > t\},$$

so, when the deaths are independent, we have $\mathbf{P}\{T(U) > t\} = \prod_{i=1}^{m} \mathbf{P}\{T(x_i) > t\}$.

The last-survivor status is denoted by $U := \overline{x_1 : \ldots : x_m}$ and fails upon the last death, and exists as long as at least one member of a group is alive, i.e.,

$$T(U) = \max(T(x_1), \ldots, T(x_m)).$$

Similarly,

$$\mathbf{P}\{T(U) \le t\} = \mathbf{P}\{\max(T(x_1), \ldots, T(x_m)) \le t\} = \mathbf{P}\{T(x_1) \le t, \ldots, T(x_m) \le t\},$$

and in the case of independent deaths, we have $\mathbf{P}\{T(U) \le t\} = \prod_{i=1}^{m} \mathbf{P}\{T(x_i) \le t\}$.

We give other statuses used in practice. Consider the general k-survivor status, which is denoted

$$U := \frac{k}{x_1 : \ldots : x_m}$$

and exists as long as at least alive k among m individuals $(x_1), \ldots, (x_m)$, i.e., it is considered destroyed upon the occurrence of the $(m-k+1)$ deaths. It is understood that the joint-life status $(k = m)$ and last-survivor status $(k = 1)$ are the special cases of the k-survivor status. Also, separately consider the $[k]$-deferred survivor status

$$U := \frac{[k]}{x_1 : \ldots : x_m}$$

and there, if alive exactly k of m individuals $(x_1), \ldots, (x_m)$, i.e., it starts at the $(m-k)$-th death and lasts until the $(m-k+1)$-th death. This status is widely used in the calculation sequences payments of limited duration [20]. Note that the new statuses can be defined by compounding. A compound status is said to exist if the status is a a combination of statuses, and at least one of them is itself a status with more than one individual. Consider, for example, some compound statuses.

- **The status** $((\overline{x_1 : x_2} : \overline{x_3 : x_4})$

  This status persists if alive at least one of (x_1) and (x_2) and at least one of (x_3) and (x_4). The time-until-failure of the status $(\overline{x_1 : x_2} : \overline{x_3 : x_4})$ is

  $$T(U) = \min\{\max\{T(x_1), T(x_2)\}, \max\{T(x_3), T(x_4)\}\}.$$

- **The status** $\left(\overline{\overline{x_1 : x_2} : (x_3 : x_4)}\right)$

  Such condition persists, if alive at least two of four, namely, (x_3) and (x_4), or when only one alive, and that either (x_1), or (x_2). The time-until-failure of the status $\left(\overline{\overline{x_1 : x_2} : (x_3 : x_4)}\right)$ is

  $$T(U) = \max\{\max\{T(x_1), T(x_2)\}, \min\{T(x_3), T(x_4)\}\}.$$

- **The status** $(x_1 : x_2 : \overline{x_3 : x_4})$

  The condition persists, if alive (x_1), (x_2) and when one is alive, and it is either (x_3), or (x_4). The time-until-failure of the status $(x_1 : x_2 : \overline{x_3 : x_4})$ is

  $$T(U) = \min\{T(x_1), T(x_2), \max\{T(x_3), T(x_4)\}\}.$$

Similarly, the fracture point may be found for the combination any statuses.

3 Functionals of the Net Premiums in Collective Life Insurance

Reasoning as in the derivation of formula (1)), in the case of the m insureds, the functionals of the net premiums in collective life insurance can be written as

$$\overline{A}_{x_1:\ldots:x_m} = \frac{\int_0^\infty e^{-\delta t}\, I_{x_1:\ldots:x_m}(t>0)dF_{x_1:\ldots:x_m}(t)}{S_{x_1:\ldots:x_m}(0)}, \tag{5}$$

$$\overline{A}_{\overline{x_1:\ldots:x_m}} = \frac{\int_0^\infty e^{-\delta t}\, I_{\overline{x_1:\ldots:x_m}}(t>0)dF_{\overline{x_1:\ldots:x_m}}(t)}{S_{\overline{x_1:\ldots:x_m}}(0)},$$

where

$$F_{x_1:\ldots:x_m}(t) = \mathbf{P}(\min\,(T(x_1),\ldots,T(x_m)) \le t)$$

and

$$F_{\overline{x_1:\ldots:x_m}(t)} = \mathbf{P}(\max\,(T(x_1),\ldots,T(x_m)) \le t)$$

are the distribution functions of the random variables $\min\,(T(x_1),\ldots,T(x_m))$ and $\max\,(T(x_1),\ldots,T(x_m))$,

$$S_{x_1:\ldots:x_m}(t) = 1 - F_{x_1:\ldots:x_m}(t) = \mathbf{P}(\min\,(T(x_1),\ldots,T(x_m) > t))$$

and

$$S_{\overline{x_1:\ldots:x_m}}(t) = 1 - F_{\overline{x_1:\ldots:x_m}}(t) = \mathbf{P}(\max\,(T(x_1),\ldots,T(x_m) > t))$$

are the corresponding survival functions.

Consider the random variables $Z_i = X_i - x_i$, $i = \overline{1,m}$. We order them in ascending and obtain the order statistics $Z_{(i)}$, $i = \overline{1,m}$. Note that the survival function

$$S_{x_1:\ldots:x_m}(0) = \mathbf{P}(\min\,(T(x_1),\ldots,T(x_m) > 0)) = \mathbf{P}(T(x_1) > 0,\ldots,T(x_m) > 0) =$$

$$= \mathbf{P}(X_1 > x_1,\ldots,X_m > x_m) = S(x_1,\ldots,x_m).$$

Then

$$\overline{A}_{x_1:\ldots:x_m} = \frac{-\int_0^\infty e^{-\delta t} I(Z_{(1)} > 0)d[1 - \mathbf{P}\{Z_{(1)} > t\}]}{\mathbf{P}\{Z_{(1)} > 0\}} =$$

$$= \frac{-\int_0^\infty e^{-\delta t} I(\min\,(T(x_1),\ldots,T(x_m)) > 0)d[1 - \mathbf{P}\{\min\,(T(x_1),\ldots,T(x_m)) > t\}]}{\mathbf{P}\{\min\,(T(x_1),\ldots,T(x_m)) > 0\}} =$$

$$= \frac{-\int_0^\infty e^{-\delta t} \prod_{j=1}^m I(T(x_j) > 0)d[1 - \mathbf{P}\{(T(x_1) > t,\ldots,T(x_m) > t)\}]}{\mathbf{P}\{T(x_1) > 0,\ldots,T(x_m) > 0\}} =$$

$$= \frac{\Phi(x_1 : \ldots : x_m, \delta)}{S(x_1, \ldots, x_m)} \tag{6}$$

and

$$\overline{A}_{\overline{x_1:\ldots:x_m}} = \frac{\int_0^\infty e^{-\delta t} I(Z_{(m)} > 0) d\mathbf{P}\{Z_{(m)} \le t\}}{\mathbf{P}\{Z_{(m)} > 0\}}.$$

By analogy with formula (1), we have

$$\overline{A}_{\underline{\quad\quad k}_{x_1:\ldots:x_m}} = \frac{\int_0^\infty e^{-\delta t} I_{Z_{(m-k+1)}}(t > 0) d\mathbf{P}\{Z_{(m-k+1)} \le t\}}{\mathbf{P}\{Z_{(m-k+1)} > 0\}}, \tag{7}$$

where $I_{Z_{(m-k+1)}}(t > 0) = I(Z_{(m-k+1)} > 0)$.

In the case of the $[k]$-deferred survivor status

$$\mathbf{P}\left\{Z_{(m-k)} < t < Z_{(m-k+1)}\right\} = \mathbf{P}\left\{t < Z_{(m-k+1)}\right\} - \mathbf{P}\left\{t < Z_{(m-k)}\right\} =$$

$$= 1 - \mathbf{P}\left\{Z_{(m-k+1)} \le t\right\} - 1 + \mathbf{P}\left\{Z_{(m-k)} \le t\right\} = \mathbf{P}\left\{Z_{(m-k)} \le t\right\} - \mathbf{P}\left\{Z_{(m-k+1)} \le t\right\},$$

and the net premium is given by the formula

$$\overline{A}_{\underline{\quad\quad [k]}_{x_1:\ldots:x_m}} = \frac{\int_0^\infty e^{-\delta t} I(Z_{(m-k)} > 0) d\mathbf{P}\{Z_{(m-k)} \le t\}}{\mathbf{P}\{Z_{(m-k)} > 0\}} -$$

$$- \frac{\int_0^\infty e^{-\delta t} I(Z_{(m-k+1)} > 0) d\mathbf{P}\{Z_{(m-k+1)} \le t\}}{\mathbf{P}\{Z_{(m-k+1)} > 0\}} = \overline{A}_{\underline{\quad\quad k-1}_{x_1:\ldots:x_m}} - \overline{A}_{\underline{\quad\quad k}_{x_1:\ldots:x_m}}.$$

The functionals of net premiums for compound statuses can be written in the same way.

4 Estimators of the Net Premiums in Collective Life Insurance

Let $(Z_{11}, \ldots, Z_{m1})$, $\ldots$, $(Z_{1n}, \ldots, Z_{mn})$ be an m-dimensional random sample and $(Z_{(1)1}, \ldots, Z_{(m)1})$, $\ldots$, $(Z_{(1)n}, \ldots, Z_{(m)n})$ be corresponding ordered set.

According to (6) as the estimator of the survival function $\mathbf{P}\left\{Z_{(1)} > t\right\}$, we take $\frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{m} I\left(Z_{ji} > t\right)$. Let $\overline{\delta}(t)$ be the Dirac function. Then, the nonparametric estimator of net premium (6) is given by

$$\hat{\overline{A}}_{x_1:\ldots:x_m} = \frac{-\int_0^\infty e^{-\delta t} \prod_{j=1}^{m} I_{Z_j}(t > 0) d[1 - \mathbf{P}_n\{Z_1 > t, \ldots, Z_m > t\}]}{\mathbf{P}_n\{Z_1 > 0, \ldots, Z_m > 0\}} =$$

$$= \frac{\int_0^\infty e^{-\delta t} \prod_{j=1}^m I_{Z_j}(t>0) \left(\frac{1}{n} \sum_{i=1}^n \prod_{j=1}^m I(Z_{ji} > t) \right)' dt}{\frac{1}{n} \sum_{i=1}^n \prod_{j=1}^m I\left(Z_{ji} > 0\right)} =$$

$$= \frac{1}{\frac{1}{n} \sum_{i=1}^n \prod_{j=1}^m I\left(Z_{ji} > 0\right)} \frac{1}{n} \sum_{i=1}^n \int_0^\infty e^{-\delta t} \prod_{j=1}^m I_{Z_j}(t>0) \overline{\delta}(t - \prod_{j=1}^m Z_{ji})\, dt =$$

$$= \frac{\frac{1}{n} \sum_{i=1}^n e^{-\delta \prod_{j=1}^m Z_{ji}} \prod_{j=1}^m I(Z_{ji} > 0)}{\frac{1}{n} \sum_{i=1}^n \prod_{j=1}^m I\left(Z_{ji} > 0\right)}. \tag{8}$$

As estimators of the distribution function $\mathbf{P}\left\{Z_{(m-k+1)} \le t\right\}$ and the survival function $\mathbf{P}\left\{Z_{(m-k+1)} > 0\right\}$, we take $\frac{1}{n} \sum_{i=1}^n I\left(Z_{(m-k+1)i} < t\right)$ and $\frac{1}{n} \sum_{i=1}^n I\left(Z_{(m-k+1)i} > 0\right)$, respectively. So, the nonparametric estimator of (4) has the form

$$\hat{\overline{A}}_{\underset{x_1:\ldots:x_m}{\underline{\quad\quad k}}} = \frac{\int_0^\infty e^{-\delta t} I_{Z_{(m-k+1)}}(t>0) \left(\frac{1}{n} \sum_{i=1}^n I(Z_{(m-k+1)i} \le t) \right)' dt}{\mathbf{P}_n\{Z_{(m-k+1)} > 0\}} =$$

$$= \frac{1}{\mathbf{P}_n\{Z_{(m-k+1)} > 0\}} \frac{1}{n} \sum_{i=1}^n \int_0^\infty e^{-\delta t} I_{Z_{(m-k+1)}}(t>0) \overline{\delta}(t - Z_{(m-k+1)i})\, dt =$$

$$= \frac{\frac{1}{n} \sum_{i=1}^n e^{-\delta \left(Z_{(m-k+1)i}\right)} I(Z_{(m-k+1)i} > 0)}{\frac{1}{n} \sum_{i=1}^n I\left(Z_{(m-k+1)i} > 0\right))}. \tag{9}$$

In the case of the $[k]$-deferred survivor status the nonparametric plug-in estimator of the net premium can be defined in the following way:

$$\hat{\overline{A}}_{\underset{x_1:\ldots:x_m}{\underline{\quad\quad [k]}}} = \hat{\overline{A}}_{\underset{x_1:\ldots:x_m}{\underline{\quad\quad k-1}}} - \hat{\overline{A}}_{\underset{x_1:\ldots:x_m}{\underline{\quad\quad k}}}.$$

5 Asymptotics of the Functions of Statistics

Introduce the notation according to [12, 6]: the function $H(t) : R^s \to R^1$, where $t = t(x) = (t_1(x), \ldots, t_s(x))$ is s-dimensional bounded function; $H_j(t) = \frac{\partial H(t)}{\partial t_j}$,

$j = \overline{1,s}$, $\nabla H(t) = (H_1(t), \ldots, H_s(t))$; the symbol T denotes the transpose; $t_n = (t_{1n}, \ldots, t_{sn})$ is s-dimensional statistic, $t_{jn} = t_{jn}(x) = t_{jn}(x, X_1, \ldots, X_n)$, $j = \overline{1,s}$; $\|t_n\| = \sqrt{t_{1n}^2 + \ldots + t_{sn}^2}$ is the Euclidean norm of t_n; $\Longrightarrow N_s\{\mu, \sigma\}$ is the symbol of weak convergence of sequence of random variables to the s-dimensional normal random variable with mean $\mu = (\mu_1, \ldots, \mu_s)$ and symmetric covariance matrix $\sigma = ||\sigma_{ij}||$, $0 < \sigma_{jj} = \sigma_{jj}(x) < \infty$, $j = \overline{1,s}$; $\Re$ is the set of integers.

D e f i n i t i o n. The function $H(t) : R^s \to R^1$ and the sequence $\{H(t_n)\}$ are said to belong to the class $\mathcal{N}_{\nu,s}(t;\gamma)$, provided that

1) there exists an ε-neighborhood $\{z : |z_i - t_i| < \varepsilon;\ i = \overline{1,s}\}$, in which the function $H(z)$ and all its partial derivatives $\dfrac{\partial H(z)}{\partial z_j}$ up to the order ν are continuous and bounded;

2) for any values of variables $X_1, ..., X_n$ the sequence $\{H(t_n)\}$ is dominated by a numerical sequence $C_0 d_n^{\gamma}$, such that $d_n \uparrow \infty$, as $n \to \infty$, and $0 \le \gamma < \infty$.

Theorem 1 [6]. Let the conditions

1) $H(z), \{H(t_n)\} \in \mathcal{N}_{2,s}(t,\gamma)$,

2) $\mathbf{E}||t_n - t||^i = O\left(d_n^{-i/2}\right)$

hold for all $i \in \Re$. Then, for every $k \in \Re$

$$\left|\mathbf{E}\left[H(t_n) - H(t)\right]^k - \mathbf{E}\left[\nabla H(t)(t_n - t)^T\right]^k\right| = O\left(d_n^{-(k+1)/2}\right). \tag{10}$$

If in formula (10) $k = 1$, we obtain the principal term $\mathbf{E}\left[\nabla H(t)(t_n - t)^T\right]$ of the bias $\mathbf{E}\left[H(t_n) - H(t)\right]$ for $H(t_n)$, and at $k = 2$, we have the principal term $\mathbf{E}\left[\nabla H(t)(t_n - t)^T\right]^2$ of the mean squared error (MSE) $\mathbf{E}\left[H(t_n) - H(t)\right]^2$.

Theorem 2 (The usual central limit theorem) [1]. If $\xi_1, ..., \xi_n, ...$ is a sequence of independent and identically distributed s-dimensional vectors,

$$\mathbf{E}\xi_k = 0, \quad \sigma(x) = \mathbf{E}\{\xi_k^T \xi_k\}, \quad t_n = \frac{1}{n}\sum_{k=1}^{n} \xi_k,$$

then, as $n \to \infty$,

$$\sqrt{n} t_n \Longrightarrow N_s\{0, \sigma(x)\}.$$

Theorem 3 [6]. If $q_n(t_n - t) \Longrightarrow N_s\{\mu, \sigma\}$ for some number sequence $q_n \uparrow \infty$, the function $H(z)$ is differentiable at the point μ, $\nabla H(\mu) \neq 0$, then

$$q_n\left(H(t_n) - H(\mu)\right) \Longrightarrow N_1\{\nabla H(\mu)\,\mu^T,\ \nabla H(\mu)\,\sigma\,\nabla H^T(\mu)\}.$$

6 Bias and MSE of Estimator $\hat{\overline{A}}_{x_1:\ldots:x_m}$

Here, we will obtain the principal term of the asymptotic MSE and the bias convergence rate of estimator (3).

Theorem 4. If the survival function $S(x_1, \ldots, x_m) > 0$ and $S(t_1, \ldots, t_m)$ is continuous at a point $(x_1, \ldots, x_m)$, then

1) for the bias $b\left(\hat{\overline{A}}_{x_1:\ldots:x_m}\right)$ of estimator (3) we have

$$\left|b\left(\hat{\overline{A}}_{x_1:\ldots:x_m}\right)\right| = \left|\mathbf{E}\hat{\overline{A}}_{x_1:\ldots:x_m} - \overline{A}_{x_1:\ldots:x_m}\right| = O\left(n^{-1}\right);$$

2) the MSE $u^2\left(\hat{\overline{A}}_{x_1:\ldots:x_m}\right)$ is given by the formula

$$u^2(\hat{\overline{A}}_{x_1:\ldots:x_m}) = \mathbf{E}\left(\hat{\overline{A}}_{x_1:\ldots:x_m} - \overline{A}_{x_1:\ldots:x_m}\right)^2 =$$

$$= \frac{\Phi(x_1,\ldots,x_m,2\delta)\,S(x_1,\ldots,x_m) - \Phi^2(x_1,\ldots,x_m,\delta)}{n\,S^3(x_1,\ldots,x_m)} + O\left(\frac{1}{n^{3/2}}\right).$$

Proof. For estimator $\hat{\overline{A}}_{x_1:\ldots:x_m}$ (3) in the notation of Theorem 1 we have:

$$s = 2, \quad t_n = (t_{1n}, t_{2n}) = (\Phi_n(x_1,\ldots,x_m,\delta), S_n(x_1,\ldots,x_m)),$$

$$d_n = n, \quad H(t_n) = \frac{\Phi_n(x_1,\ldots,x_m,\delta)}{S_n(x_1,\ldots,x_m)} = \hat{\overline{A}}_{x_1:\ldots:x_m},$$

$$t = (t_1, t_2) = (\Phi(x_1,\ldots,x_m,\delta), S(x_1,\ldots,x_m)), \quad H(t) = \frac{t_1}{t_2} = \overline{A}_{x_1:\ldots:x_m},$$

$$H_1(t) = \frac{1}{S(x_1,\ldots,x_m)}, \quad H_2(t) = -\frac{\Phi(x_1,\ldots,x_m,\delta)}{S^2(x_1,\ldots,x_m)}, \quad \nabla H(t) = (H_1(t), H_2(t)) \neq 0.$$

.

The sequence $\{H(t_n)\}$ satisfies the condition 1) of Theorem 1 with $C_0 = 1$ and $\gamma = 0$. Indeed, according to (3)

$$H(t_n) = \frac{\Phi_n(x_1,\ldots,x_m,\delta)}{S_n(x_1,\ldots,x_m)} = \frac{\frac{1}{n}\sum_{i=1}^{n} e^{-\delta\prod_{j=1}^{m} Z_{ji}} \prod_{j=1}^{m} I(Z_{ji} > 0)}{\frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{m} I\left(Z_{ji} > 0\right)} \leq 1. \tag{11}$$

Further, in view of $t_2 = S(x_1,\ldots,x_m) > 0$ the function $H(t)$ satisfies the condition 1) of Theorem 1 . Also, this function satisfies the condition 2) of Theorem 1 due to Lemma 3.1 [5], as for all $i \in \Re$ such inequalities hold:

$$\mathbf{E}\prod_{j=1}^{m} I^i\left(Z_j > 0\right) = S(x_1,\ldots,x_m) \leq 1,$$

$$\mathbf{E}e^{-i\delta\prod_{j=1}^{m} Z_j}\prod_{j=1}^{m} I^i(Z_j > 0) \leq S(x_1,\ldots,x_m) \leq 1.$$

Therefore,

$$\mathbf{E}|\Phi_n(x_1,\ldots,x_m,\delta) - \Phi(x_1,\ldots,x_m,\delta)|^i = O\left(n^{-\frac{i}{2}}\right),$$

$$\mathbf{E}|S_n(x_1,\ldots,x_m) - S_n(x_1,\ldots,x_m)|^i = O\left(n^{-\frac{i}{2}}\right).$$

Taking $k=1$ in formula (10), we get

$$\mathbf{E}\{\hat{\overline{A}}_{x_1:\ldots:x_m}\} = \overline{A}_{x_1:\ldots:x_m} + \frac{1}{t_2}\,\mathbf{E}\{t_{1n}-t_1\} - \frac{t_1}{t_2^2}\,\mathbf{E}\{t_{2n}-t_2\} + O\left(\frac{1}{n}\right).$$

Since functions $t_1 = t_1(x_1,\ldots,x_m)$, $t_2 = t_2(x_1,\ldots,x_m)$ are continuous, we have $\mathbf{E}\{t_{1n}\} = t_1$, $\mathbf{E}\{t_{2n}\} = t_2$, and $\mathbf{E}\{\hat{\overline{A}}_{x_1:\ldots:x_m}\} = \overline{A}_{x_1:\ldots:x_m} + O\left(\frac{1}{n}\right)$, i.e., $\hat{\overline{A}}_{x_1:\ldots:x_m}$ is the asymptotically unbiased estimator.

Now, putting $k=2$ in (10) and taking into account unbiasedness of t_{1n}, t_{2n}, we find the formulas for the variances and covariance:

$$u^2(\hat{\overline{A}}_{x_1:\ldots:x_m}) = \frac{1}{t_2^2}\,\mathbf{D}\{t_{1n}\} + \frac{t_1^2}{t_2^4}\,\mathbf{D}\{t_{2n}\} - 2\,\frac{t_1}{t_2^3}\,\mathrm{cov}\{t_{1n},t_{2n}\} + O\left(\frac{1}{n^{3/2}}\right). \tag{12}$$

Denote

$$\phi_i(x_1,\ldots,x_m,\delta) = e^{-\delta\,\prod_{j=1}^m Z_{ji}} \prod_{j=1}^m I(Z_{ji}>0),$$

$s_i(x_1,\ldots,x_m) = \prod_{j=1}^m I\left(Z_{ji}>0\right)$. In view of the randomness of the sample $(Z_{11},\ldots,Z_{m1})$, $\ldots$, $(Z_{1n},\ldots,Z_{mn})$, we have

$$\mathbf{D}\{t_{1n}\} = \mathbf{D}\left\{\frac{1}{n}\sum_{i=1}^n \phi_i(x_1,\ldots,x_m,\delta)\right\} = \frac{1}{n}\,\mathbf{D}\left\{\phi_1(x_1,\ldots,x_m,\delta)\right\} =$$

$$= \frac{1}{n}\left(\mathbf{E}\left\{\phi_1(x_1,\ldots,x_m,2\delta)\right\} - \mathbf{E}^2\left\{\phi_1(x_1,\ldots,x_m,\delta)\right\}\right) =$$

$$= \frac{1}{n}\left(\Phi(x_1,\ldots,x_m,2\delta) - \Phi^2(x_1,\ldots,x_m,\delta)\right),$$

$$\mathbf{D}\{t_{2n}\} = \mathbf{D}\left\{\frac{1}{n}\sum_{i=1}^n s_i(x_1,\ldots,x_m)\right\} = \frac{1}{n}\,S(x_1,\ldots,x_m)\left(1-S(x_1,\ldots,x_m)\right),$$

$$\mathrm{cov}\{t_{1n},t_{2n}\} = \frac{1}{n}\mathrm{cov}\left\{\phi_1(x_1,\ldots,x_m,\delta), s_1(x_1,\ldots,x_m))\right\} =$$

$$= \frac{1}{n}\Phi(x_1,\ldots,x_m,\delta)\left(1-S(x_1,x_2,\ldots,x_m)\right).$$

Now, we substitute the found expressions in (11) and the second assertion of the Theorem 4 has been proved.

7 Asymptotic Normality of Estimator $\hat{\overline{A}}_{x_1:\ldots:x_m}$

Theorem 5. Under the conditions of Theorem 4

$$\sqrt{n}[\hat{\overline{A}}_{x_1:\ldots:x_m} - \overline{A}_{x_1:\ldots:x_m}] \Longrightarrow$$

$$\Longrightarrow \mathsf{N}_1\left\{0, \frac{\Phi(x_1,\ldots,x_m,2\delta)\,S(x_1,\ldots,x_m) - \Phi^2(x_1,\ldots,x_m,\delta)}{S^3(x_1,\ldots,x_m)}\right\}.$$

Proof. In the notation of Theorem 2 and Theorem 3, we have: $q_n = \sqrt{n}$. Since $S(x_1,\ldots,x_m) > 0$, function $H(t) \in \mathcal{N}_{1,2}(t)$. Taking into account unbiasedness of $S_n(x_1,\ldots,x_m)$ and $\Phi_n(x_1,\ldots,x_m,\delta)$, we have $\mu^T = 0$.

According to Section 6, the elements of the covariance matrix σ are defined by the formulas:

$$\sigma_{11} = \Phi(x_1,\ldots,x_m,2\delta) - \Phi^2(x_1,\ldots,x_m,\delta),$$

$$\sigma_{12} = \sigma_{21} = \Phi(x_1,\ldots,x_m,\delta)(1 - S(x_1,\ldots,x_m)),$$

$\sigma_{22} = S(x_1,\ldots,x_m)(1 - S(x_1,\ldots,x_m))$. That is why $\nabla H(t)\,\mu^T = 0$,

$$\nabla H(t)\,\sigma\,\nabla H(t)^T = \frac{\Phi(x_1,\ldots,x_m,2\delta)\,S(x_1,\ldots,x_m) - \Phi^2(x_1,\ldots,x_m,\delta)}{S^3(x_1,\ldots,x_m)}.$$

Theorem 5 is proved.

8 Synthesis of Nonparametric Estimators of the Net Premiums in Collective Life Insurance for Other Forms of Insurance

The above considered estimators of the net premiums were constructed for whole insurance; now we will consider other forms of insurances.

- **The p-years term life insurance**

  In this case the benefit to pay if the insured will die during of the contract validity. The company does not pay the benefit if the insured has lived p years. Then

$$\hat{\overline{A}}_{\overline{x_1:\ldots:x_m}^{[k]}:p]} = \frac{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} e^{-\delta\, Z_{(m-k+1)i}} I(0 < Z_{(m-k+1)i} \le p)}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} I(Z_{(m-k+1)i} > 0)}.$$

- **The p-years endowment life insurance**

  Such form of insurance provides for a payment either following the death of the insured or upon his survival to the end of the p-years term. The given form of

insurance accumulates the client's capital. Then, the nonparametric estimator of the net premium is

$$\hat{\overline{A}}^{s}_{\overline{x_1:\ldots:x_m}^{[k]}:p]} = \frac{S_n(x_1,\ldots,x_m) - S_n(x_1+p,\ldots,x_m+p)}{S_n(x_1,\ldots,x_m)} \times$$

$$\times \hat{\overline{A}}_{\overline{x_1:\ldots:x_m}^{[k]}:p]} + \frac{S_n(x_1+p,\ldots,x_m+p)}{\overline{S}_n(x_1,\ldots,x_m)}\, e^{-\delta\, p}.$$

- **The r-years deferred life insurance**

  This form of insurance provides for a benefit following the death of the insured when he dies at least r years following policy issue. Here the net premium is expressed in the form

$$_{r|}\hat{\overline{A}}_{\overline{x_1:\ldots:x_m}^{[k]}} = \frac{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} e^{-\delta\, Z_{(m-k+1)i}} I(r < Z_{(m-k+1)i})}{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} I(Z_{(m-k+1)i} > 0)}.$$

9 Estimation of Joint-Life Annuity

As in the case of individual insurance [4, 30], we determine the joint-life annuity by making use of the corresponding net premium (see formulas (2)–(4), (6)):

$$\overline{a}_{x_1:\ldots:x_m}(\delta) = \frac{1}{\delta}\left(1 - \overline{A}_{x_1:\ldots:x_m}\right) = \frac{1}{\delta}\left(1 - \frac{\Phi(x_1:\ldots:x_m,\delta)}{S(x_1,\ldots,x_m)}\right). \tag{13}$$

So, in accordance with (3), we obtain the following estimator of the joint-life annuity:

$$\hat{\overline{a}}_{x_1:\ldots:x_m}(\delta) = \frac{1}{\delta}\left(1 - \frac{\Phi_n(x_1:\ldots:x_m,\delta)}{S_n(x_1,\ldots,x_m)}\right) =$$

$$= \frac{1}{\delta}\left(1 - \frac{\displaystyle\sum_{i=1}^{n} e^{-\delta\, \Pi_{j=1}^{m} Z_{ji}} \prod_{j=1}^{m} I(Z_{ji} > 0)}{\displaystyle\sum_{i=1}^{n}\prod_{j=1}^{m} I\left(Z_{ji} > 0\right)}\right). \tag{14}$$

Find the principal term of the asymptotic MSE and the bias convergence rate of estimator (13).

Theorem 6. If the survival function $S(x_1,\ldots,x_m) > 0$ and $S(t_1,\ldots,t_m)$ is continuous at a point $(x_1,\ldots,x_m)$, then

1) for the bias $b\left(\hat{\overline{a}}_{x_1:\ldots:x_m}(\delta)\right)$ of estimator (13) we have

$$\left|b\left(\hat{\overline{a}}_{x_1:\ldots:x_m}(\delta)\right)\right| = \left|\mathbf{E}\hat{\overline{a}}_{x_1:\ldots:x_m}(\delta) - \overline{a}_{x_1:\ldots:x_m}(\delta)\right| = O\left(n^{-1}\right);$$

2) the MSE $u^2\left(\hat{\bar{a}}_{x_1:\ldots:x_m}(\delta)\right)$ is given by the formula

$$u^2(\hat{\bar{a}}_{x_1:\ldots:x_m}(\delta)) = \mathbf{E}\left(\hat{\bar{a}}_{x_1:\ldots:x_m}(\delta) - \bar{a}_{x_1:\ldots:x_m}(\delta)\right)^2 =$$

$$= \frac{\Phi(x_1,\ldots,x_m,2\delta)\,S(x_1,\ldots,x_m) - \Phi^2(x_1,\ldots,x_m,\delta)}{n\,\delta^2\,S^3(x_1,\ldots,x_m)} + O\left(\frac{1}{n^{3/2}}\right).$$

Proof. For estimator $\hat{\bar{a}}_{x_1:\ldots:x_m}(\delta)$ (13) in the notation of Theorem 1 we have:

$$s = 2, \quad d_n = n, \quad t_n = (t_{1n}, t_{2n}) = (\Phi_n(x_1,\ldots,x_m,\delta),\, S_n(x_1,\ldots,x_m)),$$

$$H(t_n) = \frac{1}{\delta}\left(1 - \frac{\Phi_n(x_1,\ldots,x_m,\delta)}{S_n(x_1,\ldots,x_m)}\right) = \hat{\bar{a}}_{x_1:\ldots:x_m}(\delta),$$

$$t = (t_1,t_2) = (\Phi(x_1,\ldots,x_m,\delta),\, S(x_1,\ldots,x_m))\,,\ H(t) = \frac{1}{\delta}\left(1 - \frac{t_1}{t_2}\right) = \bar{a}_{x_1:\ldots:x_m}(\delta),$$

$$H_1(t) = \frac{1}{\delta S(x_1,\ldots,x_m)},\ \ H_2(t) = -\frac{\Phi(x_1,\ldots,x_m,\delta)}{\delta S^2(x_1,\ldots,x_m)},\ \ \nabla H(t) = (H_1(t), H_2(t)) \neq 0.$$

The sequence $\{H(t_n)\}$ satisfies the condition 1) of Theorem 1 with $C_0 = \dfrac{1}{\delta}$ and $\gamma = 0$. Taking into account (13) and the inequalities $0 \le \dfrac{\Phi_n(x_1,\ldots,x_m,\delta)}{S_n(x_1,\ldots,x_m)} \le 1$ (see (11)), we have

$$H(t_n) = \frac{1}{\delta}\left(1 - \frac{\Phi_n(x_1,\ldots,x_m,\delta)}{S_n(x_1,\ldots,x_m)}\right) \le \frac{1}{\delta}.$$

Further, the proof is carried out similarly to the proof of Theorem 4 and therefore is not given.

Theorem 7. Under the conditions of Theorem 4

$$\sqrt{n}[\hat{\bar{a}}_{x_1:\ldots:x_m}(\delta) - \bar{a}_{x_1:\ldots:x_m}(\delta)] \Longrightarrow$$

$$\Longrightarrow \mathsf{N}_1\left\{0, \frac{\Phi(x_1,\ldots,x_m,2\delta)\,S(x_1,\ldots,x_m) - \Phi^2(x_1,\ldots,x_m,\delta)}{\delta^2 S^3(x_1,\ldots,x_m)}\right\}.$$

Conclusions

The paper deals with the estimation problem of the current values of net premiums and life annuities. The asymptotic properties of the estimators are proved: unbiasedness, consistency and normality. The principal terms of the asymptotic MSEs of the proposed estimators are found. Statistical modeling within the framework of the de Moivre model shows that the quality of estimation according to empirical criterion improves with the growth of the sample size. Note that the improved estimators of net premiums and life annuities can be obtained by substituting of empirical survival functions by the smooth empirical survival functions (cf. [2, 3, 9, 11, 15, 17, 24], [27]–[29], [8, 36, 38, 40, 41, 43, 44]).

References

[1] Aase K.K. (1993). Premiums in a Dynamic Model of a Reinsurance Market. *Scandinavian Actuarial Journal*. No. **2**, pp. 134-160.

[2] Altman N., Leger C. (1995) Bandwidth Selection for Kernel Distribution Function Estimation. *J. Statist. Plann. Inference*. Vol. **46**, pp. 195-214.

[3] Azzalini A. (1981) A Note on the Estimation of a Distribution Function and Quantiles by a Kernel Method. *Biometrika*. Vol. **68**. No. **1**, pp. 326-328.

[4] Arora N., Arora P. (2014). Insurance Premium Optimization: Perspective of Insurance Seeker and Insurance Provider. *Journal of Management and Science*. Vol. **4**. No. **1**, pp. 43-53.

[5] Bloink R. (2010). Premium Financed Surprises: Cancellation of Indebtedness Income and Financed Life Insurance. *Tax Lawyer*. Vol. **63**. No. **3**, pp. 223-227.

[6] Bohnert A., Gatzert N. (2012). Analyzing Surplus Appropriation Schemes in Participating Life Insurance from the Insurer's and the Policyholder's Perspective. *Insurance: Mathematics and Economics*. Vol. **50**. No. **1**, pp. 64-78.

[7] Borovkov A.A. (1998) *Mathematical Statistics*. Gordon & Breach, New York.

[8] Bowers N.L., Gerber H.U., Hickman J.C., Jones D.A., Nesbitt C.J. (1986). *Actuarial Mathematics*. The Society of Actuaries, Itasca, Illinois.

[9] Bowman A., Hall P., Prvan T. (1998) Trust Bandwidth Selection for the Smoothing of Distribution Functions. *Biometrika*. Vol. **85**. No. **4**, pp. 799-808.

[10] Buhlmann H. (1984). The General Economic Premium Principle. *ASTIN Bulletin*. Vol. **14**, pp. 13-21.

[11] Chu I.S. (1995). Bootstrap Smoothing Parameter Selection for Distribution Function Estimation. *Math. Japon*. Vol. **41**. No. **1**, pp. 189-197.

[12] Dobrovidov A.V., Koshkin G.M. (1997). *Nonparametric Signal Estimation*. Nauka. Fizmatlit, Moskva (in Russian).

[13] Dong Y. (2011). Fair Valuation of Life Insurance Contracts under a Correlated Jump Diffusion Model. *ASTIN Bulletin*. Vol. **41**. No. **2**, pp. 429-447.

[14] Falin G.I. (2002). *Mathematical Foundations of the Theory of Life Insurance and Pension Schemes*. Ankil Publ., Moscow (in Russian).

[15] Falk M. (1983) Relative Efficiency and Deficiency of Kernel Type Estimators of Smooth Distribution Functions. *Statist. Neerlandica*. Vol. **37**, pp. 73-83.

[16] Faust R., Schmeiser H., Zemp A. (2013). A Performance Analysis of Participating Life Insurance Contracts. *Insurance: Mathematics and Economics*. Vol. **51**. No. **1**, pp. 158-171.

[17] Fuks I., Koshkin G. (2015). Smooth Recurrent Estimation of Multivariate Reliability Function. *Proceedings. The International Conference on Information and Digital Technologies (IDT 2015)*. Zilina, Slovakia, pp. 84-89.

[18] Gatzert N. (2008). Asset Management and Surplus Distribution Strategies in Life Insurance: An Examination with Respect to Risk Pricing and Risk Measurement. *Insurance: Mathematics and Economics*. Vol. **42**. No. **2**, pp. 839-849.

[19] Gatzert N., Wesker H. (2012). The Impact of Natural Hedging on a Life Insurer's Risk Situation. *Journal of Risk Finance*. Vol. **13**. No. **5**, pp. 396-423.

[20] Gerber H.U. (1980). Principles of Premium Calculation and Reinsurance. *Transactions of the 21st International Congress of Actuaries*, pp. 137-142.

[21] Gerber H. (1997). *Life Insurance Mathematics*. 3rd ed. Springer-Verlag, New York.

[22] Huang H., Lee Y.-T. (2010). Optimal Asset Allocation for a General Portfolio of Life Insurance Policies. *Insurance: Mathematics and Economics*. Vol. **46**. No. **2**, pp. 271-280.

[23] Ibragimov I.A., Khasminskii R.Z. (1981). *Statistical Estimation: Asymptotic Theory*. Springer, Berlin; New York.

[24] Jones M.C. (1990) The Performance of Kernel Density Functions in Kernel Distribution Function Estimation. *Statist. Probab. Lett.* Vol. **9**, pp. 129-132.

[25] Koshkin G.M. (1998). Nonparametric Estimation of Moments of Net Premium in Life Insurance. *Computer Data Analysis and Modeling: Proceedings of the Fifth International Conference*. Vol. **1**. BSU, Minsk, pp. 148-153.

[26] Koshkin G.M. (1999). Deviation Moments of the Substitution Estimator and Its Piecewise Smooth Approximations. *Siberian Mathematical Journal*. Vol. **40**. No. **3**, pp. 515-527.

[27] Koshkin G.M. (2014) Smooth Estimators of the Reliability Functions for Non-Restorable Elements. *Russian Physics Journal*. Vol. **57**. No. **5**, pp. 672-681.

[28] Koshkin G.M. (2015) Smooth Recurrent Estimators of the Reliability Functions. *Russian Physics Journal*. Vol. **58**. No. **7**, pp. 1018-1025.

[29] Koshkin G.M. (2016) Smooth Nonparametric Estimation of the Failure Rate Function and Its First Two Derivatives. *Russian Physics Journal*. Vol. **59**. No. **6**, pp. 833-844.

[30] Koshkin G.M. Gubina O.V. (2016). Estimation of the present values of life annuities for the different actuarial models. *Proceedings. The Second International Symposium on Stochastic Models, in Reliability Engineering, Life Science, and Operations Management.* The Institute of Electrical and Electronics Engineers, Beer Sheva, Israel, pp. 506-510.

[31] Koshkin G.M., Lopukhin Ya.N. (2001). On Estimation of Net Premium in Collective Life Insurance. *The 5th Korea-Russian International Symposium on Science and Technology. Proceedings. KORUS 2001.* Vol. **2**. Tomsk Polytechnic University, Tomsk, pp. 296-299.

[32] Koshkin G.M., Lopukhin Ya.N. (2001). Estimation of Net Premiums in Collective Models of Life Insurance. *XIth Annual International AFIR Colloquium.* Vol. **2**. Canadian Institute of Actuaries, Toronto, pp. 447-457.

[33] Koshkin G.M., Lopukhin Ya.N. (2001). Nonparametric Estimation of Net Premiums in Collective Insurance. *Computer Data Analysis and Modeling: Robustness and Computer Intensive Methods: Proceedings of the Sixth International Conference.* Vol. **1**. BSU, Minsk, pp. 236-241.

[34] Maurer R., Rogalla R., Siegelin I. (2013). Participating Payout Life Annuities: Lessons from Germany. *ASTIN Bulletin.* Vol. **43**. No. **2**, pp. 159-187.

[35] Nadaraya E.A. (1964). Some New Estimates of Distribution Function. *Theory of Probability and its Applications.* Vol. **9**. No. **3**, pp. 497-500.

[36] Reiss R.-D. (1981). Nonparametric Estimation of Smooth Distribution Functions. *Scand. J. Statist.* Vol. **8**, pp. 116-119.

[37] Rocio P., Aguilar C., Xu C. (2010). Design Life Insurance Participating Policies Using Optimization Techniques, International Journal of Innovative Computing. *Information and Control.* Vol. **6**. No. **4**, pp. 1655-1666.

[38] Sarda P. (1993) Smoothing Parameter Selection for Smooth Distribution Functions. *J. Statist. Plann. Inference Inf.* Vol. **35**, pp. 65-75.

[39] Schmeiser H., Wagner J. (2011). A Joint Valuation of Premium Payment and Surrender Options in Participating Life Insurance Contracts. *Insurance: Mathematics and Economics.* Vol. **49**. No. **3**, pp. 580-596.

[40] Shao Y. Xiang X. (1997). Some Extensions of the Asymptotics of a Kernel Estimator of a Distribution Function. *Statist. Probab. Lett.* Vol. **34**, pp. 301-308.

[41] Shirahata S. Chu I.S. (1992). Integrated Squared Error of Kernel-Type Estimator of Distribution Function. *Ann. Inst. Statist. Math.* Vol. **44**. No. **3**, pp. 579-591.

[42] Sliwinski A., Michalski T., Roszkiewicz M. (2013). Demand for Life Insurance - An Empirical Analysis in the Case of Poland. *The Geneva Papers on Risk and Insurance.* Vol. **38**, pp. 62-87.

[43] Swanepoel J.W.H. (1988). Mean Integrated Squared Error Properties and Optimal Kernels When Estimating a Distribution Function. *Comm. Statist. Theory Methods*. Vol. **17**. No. **11**, pp. 3785-3799.

[44] Una-Alvarez J., Gonzalez-Manteiga W., Cadarso-Suarez C. (2000). Kernel Distribution Function Estimation under the Koziol-Green Model. *J. Statist. Plann. Inference*. Vol. **87**, pp. 199-219.