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REGIONAL GRADIENT COMPENSATION WITH MINIMUM ENERGY

In this paper we interest to the regional gradient remediability or compensation problem with minimum energy. That is, when a system is subjected to disturbances, then one of the objectives becomes to find the optimal control which compensates regionally the effect of the disturbances of the system, with respect to the regional gradient observation. Therefore, we show how to find the optimal control ensuring the effect compensation of any known or unknown disturbance distributed only on a subregion of the geometrical evolution domain, with respect to the observation of the gradient on any given subregion of the evolution domain and this in finite time. Under convenient hypothesis, the minimum energy problem is studied using an extension of the Hilbert Uniqueness Method (HUM). Approximations, numerical simulations, appropriate algorithm, and illustrative examples are also presented.

Keywords: *gradient*; *optimal control*; *regional remediability*; *disturbance*; *efficient actuators.*

1. Introduction

The control problem of distributed parameter systems arises in engineering applications and many different contexts, which are characterized by a spatiotemporal evolution. Systems analysis consists of a set of concepts as controllability, observability, remediability,…that allows a better understanding of those systems and consequently enables to conduct them in a better way. Moreover, the analysis itself has to deal not with the whole domain, but with its specific subdomain of interest. Thus, motivated by practical applications, El Jai and Zerrik have introduced and studied the so-called regional analysis [1–5]. Such analysis aims to analyze or control a system in which an objective function is defined only on a prescribed subregion. Therefore, the system dynamics is defined in the whole the domain Ω, whilst the objective is focused on a given subregion ω, where $\omega \subset \Omega$. An extension of this study that is very important in practical applications is that of regional analysis of the gradient developed in [6–10]. This study is of great interest from a more practical and control point of view since there exist systems that cannot be controllable but gradient controllable or that cannot be observable but gradient observable or that cannot be detectable but gradient detectable and they provide a means to deal with some problem from the real world. With the same preoccupation, the regional gradient remediability and regionally efficient gradient actuators are introduced and characterized recently for linear distributed systems in [11].

In this work, we show how to find practically the optimal control (convenient regionally gradient efficient actuators) ensuring the gradient compensation regionally, based on a result of characterization obtained in our previous work [11]. In addition, it constitutes also an extension of our previous work [12] to the regional case.

This paper is organized as follows. In the second section, we start by presenting the considered problem. After, we recall the definitions of exact and weak regional gradient remediability, the notion of regional gradient efficient actuators, and a characterization which shows that the regional gradient remediability of any system may depend on the structure of the actuators and sensors.

In section 3, under a condition on the structure of the actuators and the weak regional gradient remediability hypothesis, using an extension of Hilbert Uniqueness Method (HUM), we examine the problem of gradient remediability with minimum energy regionally. Then, we give the optimal control which compensates an arbitrary disturbance.

In the last section, approximations, simulations, and numerical results are presented.

2. Considered problem, definitions, and characterization

Let Ω be an open and bounded subset of IR^n $((n=1,2,3)$ with a regular boundary $\partial\Omega$. Fix $T>0$ and let denoted by $Q=\Omega\times]0,T[$, $\Sigma=\partial\Omega\times]0,T[$. Consider the system described by the parabolic equation

$$\begin{cases} \frac{\partial y}{\partial t}(x,t)=A\,y(x,t)+Bu(t)+f(x,t) & - \ Q, \\ y(x,0)=y^0(x) & - \ \Omega, \\ y(\xi,t)=0 & - \ \Sigma, \end{cases} \tag{2.1}$$

where A is a second order linear differential operator which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$. $\left(S^*(t)\right)_{t\geq 0}$ is considered for the adjoint semi-group of $(S(t))_{t\geq 0}$. $B\in \mathrm{L}(U,X), u\in L^2(0,T;U)$ where U is a Hilbert space representing the control space and $X=H_0^1(\Omega)$ the state space. The disturbance term $f\in L^2(0,T;X)$ is generally unknown.

In system (2.1), the disturbance function f has a space support which can be, in practical applications, a part ω of the domain Ω $(\omega\subset\Omega)$. The system (2.1) admits a unique solution $y\in C\left(0,T;H_0^1(\Omega)\right)\cap C^1\left(0,T;L^2(\Omega)\right)$ [13] given by

$$y_{u,f}(t)=S(t)y^0+\int_0^t S(t-s)Bu(s)ds+\int_0^t S(t-s)f(s)ds\,.$$

For $\omega\subset\Omega$ an open subregion of Ω with a positive Lebesgue measure, we consider the operators

$$\chi_\omega:\left(L^2(\Omega)\right)^n\to\left(L^2(\omega)\right)^n, \quad \text{and} \quad \tilde{\chi}_\omega:L^2(\Omega)\to L^2(\omega),$$

$$y\to y\big|_\omega, \qquad\qquad y\to y\big|_\omega,$$

while their adjoints denoted by χ_ω^* and $\tilde{\chi}_\omega^*$ respectively, are defined by

$$\chi_\omega^*:\left(L^2(\omega)\right)^n\to\left(L^2(\Omega)\right)^n, \quad \text{and} \quad \tilde{\chi}_\omega^*:L^2(\omega)\to L^2(\Omega),$$

$$y\to\chi_\omega^* y=\begin{cases} y & \text{on } \omega, \\ 0 & \text{on } \Omega\backslash\omega, \end{cases} \qquad y\to\tilde{\chi}_\omega^* y=\begin{cases} y & \text{on } \omega, \\ 0 & \text{on } \Omega\backslash\omega. \end{cases}$$

Consider also the operator ∇ defined by

$$\nabla : H_0^1(\Omega) \to \left(L^2(\Omega)\right)^n,$$
$$y \to \nabla y = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n}\right),$$

while ∇^* its adjoint operator.

The system (2.1) is augmented by the regional output equation

$$z_{u,f}^{\omega}(t) = C\chi_\omega \nabla y_{u,f}(t), \qquad (2.2)$$

where $C \in L\left(\left(L^2(\omega)\right)^n, O\right)$, O is a Hilbert space (observation space). In the case of an gradient observation on $[0,T]$ with q sensors, we take generally $O = IR^q$. In the autonomous case, without disturbance ($f = 0$) and without control ($u = 0$), the gradient observation in ω is given by

$$z_{0,0}^{\omega}(t) = C\chi_\omega \nabla S(t) y^0,$$

it is then normal. However, if $f \neq 0$ and $u \neq 0$, the regional gradient observation is disturbed.

The problem consists to study the existence of an input operator B (actuators), with respect to the output operator C (sensors), ensuring the gradient compensation at finite time T, of any disturbance acting on the system, that is to say:

For any $f \in L^2(0,T;X)$, there exists $u \in L^2(0,T;U)$, such that

$$z_{u,f}^{\omega}(t) = C\chi_\omega \nabla S(T) y^0,$$

this is equivalent to

$$C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0,$$

where H and F are two operators defined by

$$H : L^2(0,T;U) \to X, \qquad F : L^2(0,T;X) \to X,$$
$$u \to Hu = \int_0^T S(T-s) Bu(s)\,ds, \quad \text{and} \quad f \to Ff = \int_0^T S(T-s) f(s)\,ds.$$

This leads to the following definitions.

Definition 1.

1) We say that the system (2.1) augmented by the output equation (2.2) is exactly regionally *f*-remediable in ω, if there exists a control $u \in L^2(0,T;U)$, such that

$$C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0.$$

2) We say that the system (2.1) augmented by the output equation (2.2) is weakly regionally *f*-remediable in ω on $[0,T]$, if for every $\varepsilon > 0$, there exists a control $u \in L^2(0,T;U)$ such that

$$\left\| C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff \right\|_{IR^q} < 0.$$

3) We say that the system (2.1) augmented by the output equation (2.2) is regionally exactly (resp. weakly) remediable in ω, if for every $f \in L^2(0,T;X)$ the system (2.1) – (2.2) is exactly (resp. weakly) f-remediable in ω.

We suppose that the system (2.1) is excited by p zone actuators $(\Omega_i, g_i)_{1\le i\le p}, g_i \in L^2(\Omega_i)$, $\Omega_i \subset \omega$, $\forall i = 1,\ldots,p$, in this case the control space is $U = IR^p$ and the operator B is given by

$$B : IR^p \to X,$$
$$u(t) = (u_1(t), u_2(t),\ldots,u_p(t)) \mapsto Bu = \sum_{i=1}^{p} \chi_{\Omega_i}(x) g_i(x) u_i(t).$$

Its adjoint is given by

$$B^* z = \left(\langle g_1, z\rangle_{\Omega_1}, \langle g_2, z\rangle_{\Omega_2}, \ldots, \langle g_p, z\rangle_{\Omega_p}\right)^T \in IR^p.$$

Also suppose that the output of the system (2.1) is given by q sensors $(D_i, h_i)_{1\le i\le q}, h_i \in L^2(D_i)$, being the spatial distribution, $D_i = \operatorname{supp} h_i \subset \omega$, for $i = 1,\ldots,q$ and $D_i \cap D_j = \phi$ for $i \ne j$, then the operator C is defined by

$$C : \left(L^2(\omega)\right)^n \to IR^q,$$
$$y(t) \mapsto Cy(t) = \left(\sum_{i=1}^{n} \langle h_1, y_i(t)\rangle_{D_1}, \sum_{i=1}^{n} \langle h_2, y_i(t)\rangle_{D_2}, \ldots, \sum_{i=1}^{n} \langle h_q, y_i(t)\rangle_{D_q}\right)^T,$$

its adjoint is given by C^* with for $\theta = (\theta_1, \theta_2, \ldots, \theta_q)^T \in IR^q$,

$$C^*\theta = \left(\sum_{i=1}^{q} \chi_{D_i}(x)\theta_i h_i(x), \sum_{i=1}^{q} \chi_{D_i}(x)\theta_i h_i(x), \ldots, \sum_{i=1}^{q} \chi_{D_i}(x)\theta_i h_i(x)\right) \in \left(L^2(\omega)\right)^n.$$

We recall the following notion of the regionally gradient efficient actuator [11].

Definition 2. The actuators $(\Omega_i, g_i)_{1\le i\le p}, g_i \in L^2(\Omega_i)$ are said to be regionally gradient efficient, if the system (2.1) – (2.2) so excited is weakly regional gradient remediable.

For $m \ge 1$, let M_m be the matrix of order $(p \times r_m)$ defined by

$$M_m = \left(\langle g_i, w_{mj}\rangle_{L^2(\Omega_i)}\right)_{ij}, 1 \le i \le p \text{ and } 1 \le j \le r_m$$

and let G_m be the matrix of order $(q \times r_m)$ defined by

$$G_m = \left(\sum_{k=1}^{n} \left\langle h_i, \frac{\partial w_{mj}}{\partial x_k}\right\rangle_{L^2(D_i)}\right)_{ij}, 1 \le i \le q \text{ and } 1 \le j \le r_m.$$

Corollary 1 [11]. If there exists $m_0 \ge 1$, such that

$$\operatorname{rank} G_{m_0}^T = q, \tag{2.3}$$

then the actuators $(\Omega_i, g_i)_{1\le i\le p}, g_i \in L^2(\Omega_i)$ are regionally gradient efficient if and only if

$$\bigcap_{m\ge 1} \ker\left(M_m G_m^T\right) = \{0\}.$$

3. Regional gradient remediability with minimum energy

Under a condition (2.3) and the weak regional gradient remediability hypothesis, we study in this section the problem of the exact regional gradient remediability with minimal energy.

For $f \in L^2(0,T;X)$, we study the existence and the unicity of an optimal control $u \in L^2(0,T;U)$ ensuring, at the time T, the exact regional gradient remediability of the disturbance f, such that

$$C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0.$$

That is the set defined by

$$D = \left\{u \in L^2\left(0,T;IR^p\right) / C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0\right\} \tag{3.1}$$

is non empty.

We consider the function

$$J(u) = \|C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff\|_{IR^q}^2 + \|u\|_{L^2\left(0,T;IR^p\right)}^2.$$

The considered problem becomes

$$\min_{u\in D} J(u).$$

For its resolution, we will use an extension of the Hilbert Uniqueness Methods (HUM).

For $\theta \in IR^q$, let us note

$$\|\theta\|_* = \left(\int_0^T \left\|B^* S^*(T-s)\nabla^* \chi_\omega^* C^* \theta\right\|_{IR^p}^2 ds\right)^{\frac{1}{2}}.$$

The corresponding inner product is given by

$$\langle\theta,\sigma\rangle_* = \int_0^T \left\langle B^* S^*(T-s)\nabla^* \chi_\omega^* C^* \theta, B^* S^*(T-s)\nabla^* \chi_\omega^* C^* \sigma\right\rangle ds$$

and the operator $\Lambda : IR^q \to IR^q$ defined by

$$\Lambda\theta = C\chi_\omega \nabla HH^* \nabla^* \chi_\omega^* C^* \theta = C\chi_\omega \nabla \int_0^T S(T-s) BB^* S^*(T-s)\nabla^* \chi_\omega^* C^* \theta\, ds.$$

Then, we have the following proposition:

Proposition 1. If the condition (2.3) is satisfied, then $\|\ \|_*$ is a norm on IR^q if and only if system (2.1) – (2.2) is weakly regional gradient remediable on $[0,T]$ and the operator Λ is invertible.

Proof. We have

$$\|\theta\|_* = \left(\int_0^T \left\|B^* S^*(T-s)\nabla^*\chi_\omega^* C^*\theta\right\|_{IR^P}^2 ds\right)^{\frac{1}{2}} = 0 \Rightarrow \left\|B^* S^*(T-s)\nabla^*\chi_\omega^* C^*\theta\right\|_{L^2(0,T;IR^P)} = 0$$

$$\Rightarrow B^* S^*(T-.)\nabla^*\chi_\omega^* C^*\theta = 0 \Rightarrow \theta \in \ker\left(B^* S^*(T-.)\nabla^*\chi_\omega^* C^*\right) = \ker\left(B^* F^*\nabla^*\chi_\omega^* C^*\right).$$

But $\ker\left(B^* F^*\nabla^*\chi_\omega^* C^*\right) = \bigcap_{m\geq 1} \ker\left(M_m f_m^\omega\right)$,

where, for $m \geq 1$,

$$f_m^\omega : \theta \in IR^q \to f_m^\omega(\theta) = \left(\left\langle\nabla^*\chi_\omega^* C^*\theta, w_{m1}\right\rangle, \left\langle\nabla^*\chi_\omega^* C^*\theta, w_{m2}\right\rangle, \ldots, \left\langle\nabla^*\chi_\omega^* C^*\theta, w_{mr_m}\right\rangle\right)^T \in IR^{r_m}.$$

Indeed, let $\theta \in IR^q$, we have

$$B^* F^*\nabla^*\chi_\omega^* C^*\theta = B^* S^*(T-.)\nabla^*\chi_\omega^* C^*\theta = \begin{pmatrix} \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta, w_{mj}\right\rangle_{L^2(\Omega)} \left\langle g_1, w_{mj}\right\rangle_{L^2(\Omega_1)} \\ \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta, w_{mj}\right\rangle_{L^2(\Omega)} \left\langle g_2, w_{mj}\right\rangle_{L^2(\Omega_2)} \\ \vdots \\ \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta, w_{mj}\right\rangle_{L^2(\Omega)} \left\langle g_P, w_{mj}\right\rangle_{L^2(\Omega_p)} \end{pmatrix}$$

and we have $\forall m \geq 1$

$$M_m f_m^\omega(\theta) = \begin{pmatrix} \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta; w_{mj}\right\rangle_{L^2(\Omega)} \left\langle g_1, w_{mj}\right\rangle_{L^2(\Omega_1)} \\ \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta; w_{mj}\right\rangle_{L^2(\Omega)} \left\langle g_2, w_{mj}\right\rangle_{L^2(\Omega_2)} \\ \vdots \\ \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta; w_{mj}\right\rangle_{L^2(\Omega)} \left\langle g_P, w_{mj}\right\rangle_{L^2(\Omega_p)} \end{pmatrix}.$$

If we assume that $\theta \in \bigcap_{m\geq 1} \ker\left(M_m f_m^\omega\right)$, then

$$\theta \in \ker\left(M_m f_m^\omega\right),\ \forall m \geq 1 \Rightarrow \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta; w_{mj}\right\rangle_{L^2(\Omega)} \left\langle g_i, w_{mj}\right\rangle = 0, \forall i \in \{1,2,\ldots,p\}, \forall m \geq 1$$

$$\Rightarrow \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \left\langle\nabla^*\chi_\omega^* C^*\theta, w_{mj}\right\rangle \left\langle g_i, w_{mj}\right\rangle = 0,\ \forall i \in \{1,2,\ldots,p\},\ \forall m \geq 1$$

$$\Rightarrow B^* F^*\nabla^*\chi_\omega^* C^*\theta = 0 \Rightarrow \theta \in \ker\left(B^* F^*\nabla^*\chi_\omega^* C^*\right),$$

where

$$\bigcap_{m\geq 1} \ker\left(M_m f_m^\omega\right) \subset \ker\left(B^* F^*\nabla^*\chi_\omega^* C^*\right),$$

that is

$$\bigcap_{m\geq 1}\ker\left(M_m f_m^{\omega}\right)=\ker\left(B^*F^*\nabla^*\chi_{\omega}^*C^*\right)$$

and we have also $\bigcap_{m\geq 1}\ker\left(M_m G_m^T\right)=\bigcap_{m\geq 1}\ker\left(M_m f_m^{\omega}\right)$. Indeed, let $\theta\in IR^q$, then

$$\theta\in\bigcap_{m\geq 1}\ker\left(M_m G_m^T\right)\ \Leftrightarrow\left(M_m G_m^T\right)\theta=0,\quad\forall m\geq 1,$$

$$\Leftrightarrow\sum_{l=1}^{q}\sum_{j=1}^{r_m}\left\langle g_i,w_{mj}\right\rangle\left\langle h_l,\sum_{k=1}^{n}\frac{\partial w_{mj}}{\partial x_k}\right\rangle_{L^2(D_l)}\theta_l=0,\forall m\geq 1,\forall i=1,\ldots,p\ ,$$

$$\Leftrightarrow\sum_{j=1}^{r_m}\left\langle g_i,w_{mj}\right\rangle\left\langle\nabla^*\chi_{\omega}^*C^*,w_{mj}\right\rangle_{L^2(\Omega)}=0,\forall m\geq 1,\forall i=1,\ldots,p\ ,$$

$$\Leftrightarrow\left(M_m f_m^{\omega}\right)\theta=0,\forall m\geq 1\ \Leftrightarrow\theta\in\bigcap_{m\geq 1}\ker\left(M_m f_m^{\omega}\right).$$

Where $\ker\left(B^*F^*\nabla^*\chi_{\omega}^*C^*\right)=\bigcap_{m\geq 1}\ker\left(M_m G_m^T\right)$, this gives $\theta\in\bigcap_{m\geq 1}\ker\left(M_m G_m^T\right)$ and since the Corollary 1, we obtain the result.

On the other hand, the operator Λ is symmetric, indeed,

$$\langle\Lambda\theta,\sigma\rangle_{IR^q}=\left\langle C\chi_{\omega}\nabla HH^*\nabla^*\chi_{\omega}^*C^*\theta,\sigma\right\rangle_{IR^q}=\left\langle\theta,C\chi_{\omega}\nabla HH^*\nabla^*\chi_{\omega}^*C^*\sigma\right\rangle_{IR^q}=\langle\theta,\Lambda\sigma\rangle_{IR^q}$$

and positive definite, indeed,

$$\begin{aligned}\langle\Lambda\theta,\theta\rangle_{IR^q}&=\left\langle C\chi_{\omega}\nabla HH^*\nabla^*\chi_{\omega}^*C^*\theta,\theta\right\rangle_{IR^q}\\&=\left\langle H^*\nabla^*\chi_{\omega}^*C^*\theta,H^*\nabla^*\chi_{\omega}^*C^*\theta\right\rangle_{L^2\left(0,T;IR^P\right)}\\&=\int_0^T\left\langle B^*S^*(T-s)\nabla^*\chi_{\omega}^*C^*\theta,B^*S^*(T-s)\nabla^*\chi_{\omega}^*C^*\theta\right\rangle_{IR^P}ds\\&=\|\theta\|_*^2>0,\quad for\ \theta\neq 0\end{aligned}$$

and then Λ is invertible. □

We give hereafter the expression of the optimal control ensuring the regional gradient remediability of a disturbance f at the time T.

Proposition 2. For $f\in L^2(0,T;X)$, there exists a unique $\theta_f\in IR^q$ such that

$$\Lambda\theta_f=-C\chi_{\omega}\nabla Ff$$

and the control

$$u_{\theta_f}(.)=B^*S^*(.)\nabla^*\chi_{\omega}^*C^*\theta_f$$

verifies

$$C\chi_{\omega}\nabla Hu_{\theta_f}+C\chi_{\omega}\nabla Ff=0\,.$$

Moreover, it is optimal and

$$\left\|u_{\theta_f}\right\|_{L^2\left(0,T;IR^P\right)}=\left\|\theta_f\right\|_*.$$

Proof. From Proposition 1, the operator Λ is invertible; then, for $f \in L^2(0,T;X)$, there exists a unique $\theta_f \in IR^q$, such that $\Lambda\theta_f = -C\chi_\omega \nabla F f$ and, if we put $u_{\theta_f}(.) = B^* S^*(.)\nabla^* \chi_\omega^* C^* \theta_f$, we obtain

$$\Lambda\theta_f = C\chi_\omega \nabla \int_0^T S(T-s) BB^* S^*(T-s)\nabla^* \chi_\omega^* C^* \theta_f \, ds = C\chi_\omega \nabla H u_{\theta_f}$$

$$\Rightarrow -C\chi_\omega \nabla F f = C\chi_\omega \nabla H u_{\theta_f} \Rightarrow C\chi_\omega \nabla H u_{\theta_f} + C\chi_\omega \nabla F f = 0 .$$

The set D defined by (4.1) is closed, convex, and not empty. For $u \in D$, we have $J(u) = \|u\|^2_{L^2(0,T;IR^P)}$. J is strictly convex on D, and then has a unique minimum at $u^* \in D$, characterized by

$$\left\langle u^*, v - u^* \right\rangle_{L^2(0,T;IR^P)} \geq 0; \quad \forall v \in D .$$

For $v \in D$, we have

$$\left\langle u_{\theta_f}, v - u_{\theta_f} \right\rangle_{L^2(0,T;IR^P)} = \left\langle B^* S^*(.)\nabla^* \chi_\omega^* C^* \theta_f, v - B^* S^*(.)\nabla^* \chi_\omega^* C^* \theta_f \right\rangle_{L^2(0,T;IR^P)}$$

$$= \left\langle \theta_f, C\chi_\omega \nabla H v - \Lambda\theta_f \right\rangle_{IR^q} = 0 .$$

Since u^* is unique, then $u^* = u_{\theta_f}$ and u_{θ_f} is optimal with

$$\left\| u_{\theta_f} \right\|^2_{L^2(0,T;IR^P)} = \left\| B^* S^*(.)\nabla^* \chi_\omega^* C^* \theta_f \right\|^2_{L^2(0,T;IR^P)} = \left\| \theta_f \right\|^2_* . \ \square$$

4. Approximations and numerical simulations

This section concerns approximations and numerical simulations of the problem of gradient remediability.

First, we give an approximation of θ_f as a solution of a finite dimension linear system $Ax = b$ and then the optimal control u_{θ_f}, with a comparison between the corresponding observation noted $z^\omega_{u_{\theta_f},f}$ and $z^\omega_{0,0}$ the observation corresponding to the autonomous case.

4.1. Approximations

Coefficients of the system: For $i, j \geq 1$, let $a_{ij} = \left\langle \Lambda e_i, e_j \right\rangle_{IR^q}$, where $(e_i)_{1 \leq i \leq q}$ is the canonical basis of IR^q, we have

$$\Lambda e_i = C\chi_\omega \nabla \int_0^T S(T-s) BB^* S^*(T-s)\nabla^* \chi_\omega^* C^* e_i \, ds .$$

And for M, N sufficiently large, we have

$$a_{ij} \approx \sum_{m=1}^{M}\sum_{l=1}^{r_m}\sum_{m'=1}^{N}\sum_{h=1}^{r_{m'}}\sum_{\tau=1}^{p}\left(\frac{e^{(\lambda_m+\lambda_{m'})T}-1}{\lambda_m+\lambda_{m'}}\right)\left\langle g_\tau, w_{ml}\right\rangle_{L^2(\Omega_\tau)}\left\langle g_\tau, w_{m'h}\right\rangle_{L^2(\Omega_\tau)}$$
$$\times\sum_{k'=1}^{n}\left\langle \frac{\partial w_{m'h}}{\partial x_{k'}}, h_i\right\rangle_{L^2(D_i)}\sum_{k=1}^{n}\left\langle \frac{\partial w_{ml}}{\partial x_k}, h_j\right\rangle_{L^2(D_j)} \quad (4.1)$$

and $\Lambda\theta_f = -C\chi_\omega \nabla Ff$, then $b_j = -\left\langle C\chi_\omega\nabla Ff, e_j\right\rangle_{IR^q}$.

For N sufficiently large, we have

$$b_j \approx -\sum_{m'=1}^{N}\sum_{h=1}^{r_{m'}}\sum_{k=1}^{n}\left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_j\right\rangle_{L^2(D_j)}\int_0^T e^{\lambda_{m'}(T-s)}\left\langle f(s), w_{m'h}\right\rangle_{L^2(\omega)} ds\,. \quad (4.2)$$

The optimal control: In this part, we give an approximation of the optimal control u_{θ_f} which is defined by $u_{\theta_f}(.) = B^* S^*(T-.)\nabla^*\chi_\omega^* C^*\theta_f$. Its function coordinates $u_{j,\theta_f}(.)$ are given by

$$u_{j,\theta_f}(.) = \left\langle g_j, S^*(T-.)\nabla^*\chi_\omega^* C^*\theta_f\right\rangle$$
$$\approx \sum_{m'=1}^{N}\sum_{h=1}^{r_{m'}}\sum_{k=1}^{n}\sum_{i=1}^{q}\theta_{i,f}e^{\lambda_{m'}(T-.)}\left\langle g_j, w_{m'h}\right\rangle_{L^2(\Omega_j)}\left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_i\right\rangle_{L^2(D_i)} \quad (4.3)$$

for a large integer N.

Cost: The minimum energy (cost) is defined by

$$\left\|u_{\theta_f}\right\|_{L^2(0,T;IR^p)} = \left(\int_0^T \left\|B^* S^*(T-s)\nabla^*\chi_\omega^* C^*\theta_f\right\|_{IR^p}^2 ds\right)^{\frac{1}{2}}$$
$$\approx\left(\sum_{j=1}^{p}\int_0^T\left(\sum_{m'=1}^{N}\sum_{h=1}^{r_{m'}}\sum_{k=1}^{n}\sum_{i=1}^{q}e^{\lambda_m(T-s)}\theta_{i,f}\left\langle h_i, \frac{\partial w_{m'h}}{\partial x_k}\right\rangle_{L^2(D_i)}\left\langle g_j, w_{m'h}\right\rangle_{L^2(\Omega_j)}\right)^2 ds\right)^{\frac{1}{2}}$$

for N sufficiently large.

The corresponding observation: The observation corresponding to the control is given by

$$z^\omega_{u_{\theta_f},f}(t) = C\chi_\omega\nabla S(t)y^0 + C\chi_\omega\nabla\int_0^t S(t-s)Bu_{\theta_f}(s)ds + C\chi_\omega\nabla\int_0^t S(t-s)f(s)ds\,. \quad (4.4)$$

Its coordinates $\left(z^\omega_{j,u_{\theta_f},f}(.)\right)_{1\le j\le q}$ are obtained for a large integer N as follows:

$$z^\omega_{j,u_{\theta_f},f}(t) \approx \sum_{m'=1}^{N}\sum_{h=1}^{r_{m'}}\sum_{k=1}^{n}e^{\lambda_{m'}t}\left\langle y^0, w_{m'h}\right\rangle_{L^2(\omega)}\left\langle h_j, \frac{\partial w_{m'h}}{\partial x_k}\right\rangle_{L^2(D_j)}$$
$$+\sum_{m'=1}^{N}\sum_{h=1}^{r_{m'}}\sum_{k=1}^{n}\sum_{i=1}^{p}\left\langle g_i, w_{m'h}\right\rangle_{L^2(\Omega_i)}\left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_j\right\rangle_{L^2(D_j)}\int_0^t e^{\lambda_{m'}(t-s)}u_{i,\theta_f}(s)ds \quad (4.5)$$
$$+\sum_{m'=1}^{N}\sum_{h=1}^{r_{m'}}\sum_{k=1}^{n}\left\langle \frac{\partial w_{m'h}}{\partial x_k}, h_j\right\rangle_{L^2(D_j)}\int_0^t e^{\lambda_{m'}(t-s)}\left\langle f(s), w_{m'h}\right\rangle_{L^2(\omega)} ds.$$

4.2. Numerical simulations

We recall the problem considered above:

$$(P)\begin{cases}\text{Find} \quad u\in L^2(0,T;U), \text{ such that}\\ C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0.\end{cases}$$

Based on Proposition 2 and using the above results, we can develop an algorithm which allows us to determine a sequence of controls which converges to the solution u^* of (P). The output is given by (4.4) – (4.5).

Algorithm

Step 1: Data: the domain Ω, the subregion ω, the final time T, the initial state y^0, the disturbance function f, the sensors (D,h), the efficient gradient actuators (σ, g), and the accuracy threshold ε.

Step 2: Choose a low truncation order $M = N$.

Step 3: Compute $z^\omega_{0,0}$: the output, when $f = 0$ and $u = 0$ (an autonomous case).

Step 4: Compute $z^\omega_{0,f}$: the output, when $f \neq 0$ and $u = 0$ (a disturbed case).

Step 5: Solve a finite dimension linear system $\mathrm{A}\theta = b$, where these coefficients are given by (4.1) – (4.2).

Step 6: Calculate u given by (4.3).

Step 7: Compute $z^\omega_{u,f}$: the output when $f \neq 0$ and $u \neq 0$ (a disturbed and controlled case, that is to say a compensate case).

Step 8: If $\left\| z^\omega_{u,f} - z^\omega_{0,0} \right\|_{L^2(\omega)} \leq \varepsilon$, then stop. Otherwise,

Step 9: $M \leftarrow M+1$ and $N \leftarrow N+1$ and return to step 3.

Step 10: the optimal control u corresponds to the solution u^* of (P).

Now, we give a numerical example, which illustrates the efficiency of the approach given in the above section.

Illustrative example. We consider without loss of generally the following diffusion system

$$\begin{cases}\dfrac{\partial y}{\partial t}(x,t) = \Delta y(x,t) + \sum_{i=1}^{p} \chi_{\Omega_i} g_i(x) u_i(t) + f(x,t) & \Omega\times]0,T[\\ y(x,0) = y^0(x) & \Omega\\ y(\xi,t) = 0 & \partial\Omega\times]0,T[\end{cases}$$

with $\Omega =]0,1[$ and a Dirichlet boundary condition. In this case, the functions $w_m(.)$ are defined by

$$w_m(x) = \sqrt{2}\sin(m\pi x); m \geq 1.$$

The associated eigenvalues are simple and given by

$$\lambda_m = -m^2\pi^2; \quad m \geq 1.$$

Let $\omega =]0.15, 0.25[$ $(\omega \subset \Omega)$ be the geometrical support or the disturbance.

Then in the case of:

- an initial state: $y^0(.) = 0$,

- a sensor: (D,h) with $D=\omega$ and $h(x)=\sqrt{2}x^2$ $(q=1)$,
- an efficient actuator: (σ,g) with $\sigma=\omega$ and $g(x)=2x^3$ $(p=1)$,
- a disturbance function: defined by $f(x)=240\cos x$.

For $M=N=10$ and $T=0.5$, we obtain numerical results illustrating the theoretical results established in previous sections. Hence, in Fig. 1, we give the representations of the discrete observation $z_{u,f}$ corresponding to the control $u=u_{\theta_f}$ and the disturbance f and $z_{0,0}$, which represent the normal observation, that is $u=0$ and $f=0$.

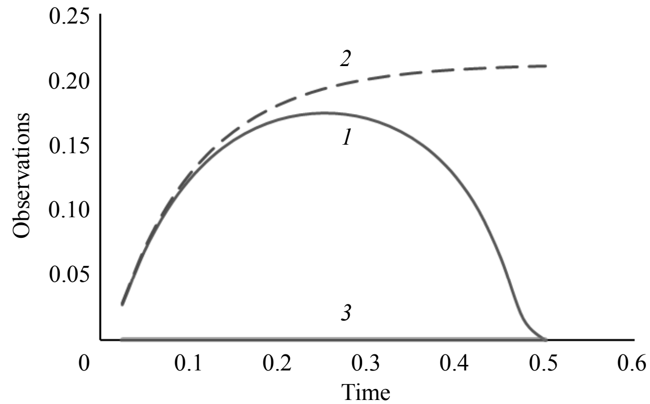

Fig. 1. Representation of $z_{u,f}$ (line *1*), $z_{0,f}$ (line *2*) and $z_{0,0}$ (line *3*)

This figure shows that the disturbance f is compensated by the optimal control $u=u_{\theta_f}$ at the time T that is, we have $z^{\omega}_{u,f}(T)=z^{\omega}_{0,0}(T)$.

The optimal control u_{θ_f} ensuring the regional gradient remediability of the disturbance f is represented in Fig. 2.

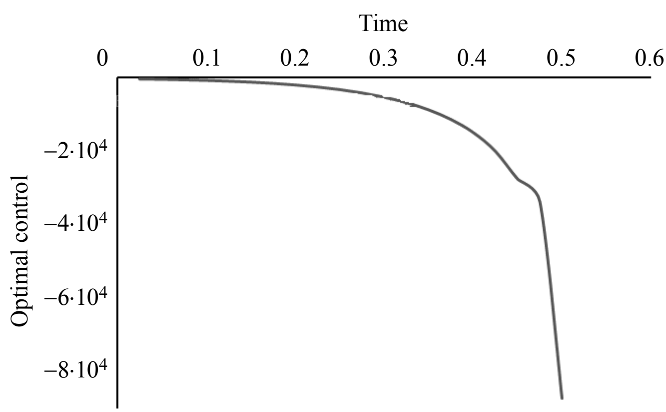

Fig. 2. Representation of the optimal control u_{θ_f}

Table 1 shows that if we want to eliminate the effect of the disturbance in a short time T , the cost increases.

Table 1

Evolution cost with respect to the finite time *T*

T	Cost
0.3	$1.06 \cdot 10^5$
0.4	$1.05 \cdot 10^5$
0.5	$1.03 \cdot 10^5$
1	$9.46 \cdot 10^4$
2	$8.99 \cdot 10^4$
3	$8.89 \cdot 10^4$
5	$8.85 \cdot 10^4$
10	$8.84 \cdot 10^4$
100	$8.84 \cdot 10^4$

Conclusions

Under a condition on the sensors and the weak regional gradient remediability hypothesis, we have studied the problem of exact regional gradient remediability with minimal energy. That is to say, when a system is subjected to disturbances, we have shown how to find the optimal control, which compensate the effect of the disturbance that can be located in a given subregion of the space domain, with respect to the regional gradient observation and this using an extension of the Hilbert Uniqueness Method. Illustrative examples, numerical approximations, and results are also presented. These results are developed for a class of discrete linear distributed parabolic systems, but the considered approach can be extended to regional or bounded gradient remediability of other class of systems with a convenient choice of space.

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Реккаб С., Айчаой Х., Бенхадид С. ЛОКАЛЬНАЯ ГРАДИЕНТНАЯ КОМПЕНСАЦИЯ ПРИ МИНИМУМЕ ЭНЕРГИИ. *Вестник Томского государственного университета. Математика и механика.* 2019. № 61. С. 19–31

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Ключевые слова: градиент, оптимальное управление, локальная восстановимость, возмущение, эффективные актюаторы.

Рассматривается проблема локальной градиентной восстановимости или компенсации при минимальных затратах энергии. Иными словами, при возмущении системы одной из задач становится отыскание оптимального управления, которое локально компенсирует результат возмущения системы по отношению к локальному градиентному измерению. Таким образом, показано, как найти оптимальное управление, обеспечивающее компенсацию любого известного или неизвестного возмущения, распределённого лишь на части области геометрического роста, по отношению к измерению градиента на любой заданной подобласти области роста за конечное время. Проблема минимума энергии исследуется при удобных предположениях с помощью обобщенного метода единственности Гильберта. Представлены также приближения, численное моделирование, соответствующий алгоритм и иллюстративный примеры.

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