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Estimation and testing of a small change location in the intensity of a Poisson process *

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Abstract

A model of Poissonian observations having a jump (change-point) in the intensity function is considered in the case when the size of the jump converges to zero. The limiting likelihood ratio in this case is quite different from the one corresponding to the case of a fixed jump-size. More precisely, we show that the limiting likelihood ratio is a log-Wiener process, and so, this model is asymptotically equivalent to the well known model of a signal in white Gaussian noise. Further, we deduce the properties of the maximum likelihood and Bayesian estimators, as well as those of the general likelihood ratio, Wald's and two Bayesian tests. We illustrate the results by numerical simulations.

Keywords: Poisson process, non-regularity, change-point, small jump, statistical estimation, hypothesis testing.

We observe n independent realizations $X_j^{(n)} = \{X_j^{(n)}(t),\ t \in [0,\tau]\}$, $j = 1,\dots,n$, of an inhomogeneous Poisson process on the interval $[0,\tau]$ (the constant $\tau > 0$ is supposed to be known) with intensity function $\lambda_\vartheta^{(n)}$, where $\vartheta \in \Theta = (\alpha,\beta)$, $0 \le \alpha < \beta \le \tau$, is some unknown parameter. The observation will be denoted $X^{(n)} = \{X_1^{(n)},\dots,X_n^{(n)}\}$, while the corresponding probability distribution and expectation will be denoted $\mathbf{P}_\vartheta^{(n)}$ and $\mathbf{E}_\vartheta^{(n)}$ respectively.

The parameter ϑ corresponds to the location of a jump in the (elsewhere continuous) intensity function $\lambda_\vartheta^{(n)}$. The size of the jump (depending on n) will be denoted r_n and will be supposed converging to some $r \in \mathbb{R}$. As we will see below, the behavior of our model depends on either one has $r \neq 0$ or $r = 0$ and is quite different in these two cases.

More precisely, we assume that the following conditions are satisfied.

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(C1) $\lambda_\vartheta^{(n)}(t) = \psi_n(t) + r_n \mathbf{1}_{\{t>\vartheta\}}$, where the function ψ_n is continuous on $[0,\tau]$.

(C2) For all $t \in [0,\tau]$, there exist the $\lim\limits_{n\to+\infty} \psi_n(t) = \psi(t) > 0$ and, moreover, this convergence is uniform with respect to t.

(C3) As $n \to +\infty$, the jump size r_n converges to some $r \in \mathbb{R}$, that is, $r_n \to r$. In the case $r = 0$, we also suppose that this convergence ($r_n \to 0$) is slower than $n^{-1/2}$, that is, $n\, r_n^2 \to +\infty$.

(C4) The family of functions $\{\lambda_\vartheta^{(n)}\}_{n\in\mathbb{N},\vartheta\in\Theta}$ is uniformly strictly positive and uniformly bounded, that is, there exist some constants $\ell, L > 0$ such that

$$\ell \le \lambda_\vartheta^{(n)}(t) \le L$$

for all $n \in \mathbb{N}$, $\vartheta \in \Theta$ and $t \in [0,\tau]$.

Likelihood ratio. The likelihood of our model (with respect to a standard Poisson process of intensity 1) is given by (see, for example, [5])

$$L_n\bigl(\vartheta, X^{(n)}\bigr) = \exp\Biggl\{\sum_{j=1}^{n} \int_{[0,\tau]} \ln \lambda_\vartheta^{(n)}(t)\; X_j^{(n)}(\mathrm{d}t) - n \int_0^\tau \bigl[\lambda_\vartheta^{(n)}(t) - 1\bigr]\, \mathrm{d}t\Biggr\}.$$

We put $\varphi_n = \frac{1}{|r|n}$ in the case $r \ne 0$ and $\varphi_n = \frac{\psi(\vartheta)}{n\, r_n^2}$ in the case $r = 0$, and we introduce the normalized likelihood ratio

$$Z_{n,\vartheta}(v) = \frac{L_n\bigl(\vartheta + v\varphi_n, X^{(n)}\bigr)}{L_n\bigl(\vartheta, X^{(n)}\bigr)},$$

where $v \in V_n = \bigl(\varphi_n^{-1}(\alpha - \vartheta), \varphi_n^{-1}(\beta - \vartheta)\bigr)$.

Note that in both cases we have (by the condition **C3** in the case $r = 0$) $\varphi_n \to 0$.

Note also that the trajectories of the process $Z_{n,\vartheta}$ are càdlàg functions. Moreover, correctly extending these trajectories to the whole real line, one can consider that they belong to the Skorohod space $\mathcal{D}_0(\mathbb{R})$. This space is defined as the space of functions f on $\mathbb{R}$ which do not have discontinuities of the second kind and which are vanishing at infinity, that is, such that $\lim\limits_{u\to\pm\infty} f(u) = 0$. We assume that all the functions $f \in \mathcal{D}_0(\mathbb{R})$ are continuous from the right (are càdlàg).

The asymptotic behavior of the normalized likelihood ratio $Z_{n,\vartheta}$ (in the sense of the weak convergence in the space $\mathcal{D}_0(\mathbb{R})$ as $n \to \infty$) depends on either one has $r \ne 0$ or $r = 0$ and is quite different in these two cases. So, the limit process must be introduced specifically in each case.

Case $r \neq 0$ limit process. In the case $r \neq 0$, the limit process is a log-Poisson type process and is introduced by

$$Z_\rho^*(v) = \begin{cases} \exp\bigl\{\rho Y^+(v) - v\bigr\}, & \text{if } v \geq 0, \\ \exp\bigl\{-\rho Y^-\bigl((-v)-\bigr) - v\bigr\}, & \text{if } v < 0, \end{cases}$$

where $\rho = \bigl|\ln \frac{\psi(\vartheta)}{\psi(\vartheta)+r}\bigr|$, and Y^+ and Y^- are independent Poisson processes on $\mathbb{R}_+$ of constant intensities $\frac{1}{e^\rho - 1}$ and $\frac{1}{1-e^{-\rho}}$ respectively.

Note that the process Z_ρ^* was studied in [1] and that its trajectories almost surely belong to the space $\mathcal{D}_0(\mathbb{R})$.

Case $r = 0$ limit process. In the case $r = 0$, the limit process is a log-Wiener type process and is introduced by

$$Z^*(v) = \exp\biggl\{W(v) - \frac{|v|}{2}\biggr\}, \quad v \in \mathbb{R},$$

where W is a double-sided Brownian motion (Wiener process).

Note that the trajectories of the processes Z^* almost surely belong to the space $\mathcal{C}_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity, and that $\mathcal{C}_0(\mathbb{R}) \subset \mathcal{D}_0(\mathbb{R})$.

Asymptotic behavior of the likelihood ratio. Now we can state the following theorem about the asymptotic behavior of the normalized likelihood ratio. The proof of this theorem (as well as those of the other results presented below) can be found in [4].

Theorem 1. *Let the conditions* **C1** – **C4** *be fulfilled. Then, the process $Z_{n,\vartheta}$ converges weakly in the space $\mathcal{D}_0(\mathbb{R})$ to*

- *the process Z_ρ^*, in the case $r < 0$,*
- *the process $Z_\rho^\star$ defined by $Z_\rho^\star(v) = Z_\rho^*\bigl((-v)-\bigr)$, in the case $r > 0$,*
- *the process Z^*, in the case $r = 0$.*

Let us note that in the case $r \neq 0$, the limiting likelihood ratio is the same as in the fixed jump-size case, and so the properties of the estimators and of the tests are also the same (see [5, 6] for the properties of the estimators and [3] for the properties of the tests). So, in the sequel, we consider the case $r = 0$ only.

Parameter estimation. Recall that, as function of ϑ, the likelihood of our model is discontinuous (has jumps). So, the maximum likelihood estimator $\widehat{\vartheta}_n$ of ϑ is introduced through the equation

$$\max\Bigl\{L_n\bigl(\widehat{\vartheta}_n+, X^{(n)}\bigr), L_n\bigl(\widehat{\vartheta}_n-, X^{(n)}\bigr)\Bigr\} = \sup_{\vartheta\in\Theta} L_n\bigl(\vartheta, X^{(n)}\bigr).$$

The Bayesian estimator $\widetilde{\vartheta}_n$ of ϑ for a given prior density p and for square loss is defined by

$$\widetilde{\vartheta}_n = \frac{\int_\alpha^\beta \vartheta\, p(\vartheta) L_n\bigl(\vartheta, X^{(n)}\bigr)\,\mathrm{d}\vartheta}{\int_\alpha^\beta p(\vartheta) L_n\bigl(\vartheta, X^{(n)}\bigr)\,\mathrm{d}\vartheta}\,.$$

We introduce the random variables ξ^* and ζ^* by the equations

$$Z^*(\xi^*) = \sup_{v\in\mathbb{R}} Z^*(v),$$

and

$$\zeta^* = \frac{\int_{-\infty}^{+\infty} v\, Z^*(v)\,\mathrm{d}v}{\int_{-\infty}^{+\infty} Z^*(v)\,\mathrm{d}v}\,.$$

Now we can state the following theorem giving an asymptotic lower bound on the risk of all the estimators of ϑ.

Theorem 2. *Let the conditions* **C1** – **C4** *be fulfilled with* $r = 0$. *Then, for any* $\vartheta_0 \in \Theta$, *we have*

$$\lim_{\delta\to 0}\ \varliminf_{n\to+\infty}\ \inf_{\overline{\vartheta}_n}\ \sup_{|\vartheta-\vartheta_0|<\delta} \varphi_n^{-2}\,\mathbf{E}_\vartheta^{(n)}(\overline{\vartheta}_n - \vartheta)^2 \geq \mathbf{E}(\zeta^*)^2,$$

where the inf *is taken over all possible estimators* $\overline{\vartheta}_n$ *of the parameter* ϑ.

This theorem allows us to introduce the following definition.

Definition 3. *Let the conditions* **C1** – **C4** *be fulfilled with* $r = 0$. *We say that an estimator* ϑ_n^* *is asymptotically efficient if*

$$\lim_{\delta\to 0}\ \lim_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\delta} \varphi_n^{-2}\,\mathbf{E}_\vartheta^{(n)}(\vartheta_n^* - \vartheta)^2 = \mathbf{E}(\zeta^*)^2$$

for all $\vartheta_0 \in \Theta$.

Now, we can state the following two theorems giving the asymptotic properties of the maximum likelihood and Bayesian estimators.

Theorem 4. *Let the conditions* **C1** – **C4** *be fulfilled with* $r = 0$. *Then the maximum likelihood estimator* $\widehat{\vartheta}_n$ *satisfies the relations*

$$\mathbf{P}_\vartheta^{(n)} - \lim_{n\to+\infty} \widehat{\vartheta}_n = \vartheta,$$

$$\mathcal{L}_\vartheta^{(n)}\bigl\{\varphi_n^{-1}(\widehat{\vartheta}_n - \vartheta)\bigr\} \Rightarrow \mathcal{L}(\xi^*)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_\vartheta^{(n)} \varphi_n^{-p}\bigl|\widehat{\vartheta}_n - \vartheta\bigr|^p = \mathbf{E}\,|\xi^*|^p \quad \textit{for any} \quad p > 0.$$

In particular, the relative asymptotic efficiency of $\widehat{\vartheta}_n$ *is* $\mathbf{E}(\zeta^*)^2/\mathbf{E}(\xi^*)^2$.

Theorem 5. *Let the conditions* **C1** – **C4** *be fulfilled with* $r = 0$. *Then, for any continuous strictly positive density, the Bayesian estimator* $\widetilde{\vartheta}_n$

satisfies the relations

$$\mathbf{P}_\vartheta^{(n)} - \lim_{n\to+\infty} \widetilde{\vartheta}_n = \vartheta,$$

$$\mathcal{L}_\vartheta^{(n)}\bigl\{\varphi_n^{-1}(\widetilde{\vartheta}_n - \vartheta)\bigr\} \Rightarrow \mathcal{L}(\zeta^*)$$

and

$$\lim_{n\to+\infty} \mathbf{E}_\vartheta^{(n)} \varphi_n^{-p} \bigl|\widetilde{\vartheta}_n - \vartheta\bigr|^p = \mathbf{E}\,|\zeta^*|^p \quad \textit{for any} \quad p > 0.$$

In particular, $\widetilde{\vartheta}_n$ *is asymptotically efficient.*

Hypothesis testing. We consider the same model of observation as above, with the only difference that now we suppose that the parameter $\theta \in \Theta = [\vartheta_0, \beta)$, $0 < \vartheta_0 < \beta \leq \tau$. We assume that the conditions **(C1)**–**(C4)** are fulfilled with $r = 0$ and we want to test the following two hypotheses:

$$\mathcal{H}_0 \ : \ \vartheta = \vartheta_0,$$

$$\mathcal{H}_1 \ : \ \vartheta > \vartheta_0.$$

We define a (randomized) test $\overline{\phi}_n = \overline{\phi}_n\bigl(X^{(n)}\bigr)$ as the probability to accept the hypothesis $\mathcal{H}_1$. The size of the test is $\mathbf{E}_{\vartheta_0}^{(n)}\overline{\phi}_n\bigl(X^{(n)}\bigr)$, and its power function is $\beta(\overline{\phi}_n, \vartheta) = \mathbf{E}_\vartheta^{(n)}\overline{\phi}_n\bigl(X^{(n)}\bigr)$, $\vartheta > \vartheta_0$. As usually, we denote $\mathcal{K}_\varepsilon$ the class of the tests of asymptotic size $\varepsilon \in [0,1]$, that is,

$$\mathcal{K}_\varepsilon = \Bigl\{\overline{\phi}_n \ : \ \lim_{n\to+\infty} \mathbf{E}_{\vartheta_0}^{(n)}\overline{\phi}_n\bigl(X^{(n)}\bigr) = \varepsilon\Bigr\}.$$

The comparison of tests is done by comparison of their power functions. However, it is known that for any reasonable test and for any fixed alternative the power function tends to 1. To avoid this difficulty, we use Pitman's approach and consider *contiguous* (or *close*) alternatives. More precisely, changing the variable $\vartheta = \vartheta_u = \vartheta_0 + u\varphi_n$, the initial problem of hypothesis testing is replaced by the following one:

$$\mathcal{H}_0 \ : \ u = 0,$$

$$\mathcal{H}_1 \ : \ u > 0,$$

and the power function is now $\beta(\overline{\phi}_n, u) = \mathbf{E}_{\vartheta_n}^{(n)}\overline{\phi}_n\bigl(X^{(n)}\bigr)$, $u > 0$.

The study is essentially based on the properties of the normalized likelihood ratio established above. Note that the limit of the normalized likelihood ratio at the point $\vartheta = \vartheta_0$ (under hypothesis $\mathcal{H}_0$) is the following:

$$Z_{n,\vartheta_0}(v) \Rightarrow Z^*(v), \quad v \geq 0.$$

Under the alternative ϑ_u (with some fixed $u > 0$), we obtain

$$Z_{n,\vartheta_0}(v) \Rightarrow Z_u^*(v) = \exp\left\{W(v) - \frac{|v-u|}{2} + \frac{u}{2}\right\}, \quad v \geq 0.$$

The score-function test — which is locally asymptotically uniformly most powerful (LAUMP) in the regular case $\big($see [2]$\big)$— does not exist in this non-regular situation. So, we construct and study the general likelihood ratio test (GLRT), Wald's test (WT) and two Bayesian tests (BT1 and BT2).

The GLRT is defined by

$$\widehat{\phi}_n\big(X^{(n)}\big) = \mathbf{1}_{\left\{Q(X^{(n)})>\frac{1}{\varepsilon}\right\}},$$

where

$$Q\big(X^{(n)}\big) = \sup_{\vartheta>\vartheta_0} \frac{L_n\big(\vartheta, X^{(n)}\big)}{L_n\big(\vartheta_0, X^{(n)}\big)}.$$

It belongs to $\mathcal{K}_\varepsilon$ and its power function has the following limit:

$$\beta\big(\widehat{\phi}_n, u\big) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\sup_{v>0} Z_{n,\vartheta_0}(v) > 1/\varepsilon\right\} \to \mathbf{P}\left\{\sup_{v>0} Z_u^*(v) > 1/\varepsilon\right\}.$$

The WT is defined by

$$\widehat{\phi}_n^\star\big(X^{(n)}\big) = \mathbf{1}_{\left\{\varphi_n^{-1}\left(\widehat{\vartheta}_n-\vartheta_0\right)>m_\varepsilon\right\}},$$

where m_ε is the solution of the following equation (here Φ is the distribution function of the standard Gaussian low):

$$\int_{m_\varepsilon}^{+\infty}\left(\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{t}{8}\right\} - \frac{1}{2}\Phi\left(-\frac{\sqrt{t}}{2}\right)\right)\mathrm{d}t = \varepsilon.$$

It belongs to $\mathcal{K}_\varepsilon$ and its power function has the following limit:

$$\beta\big(\widehat{\phi}_n^\star, u\big) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\varphi_n^{-1}\big(\widehat{\vartheta}_n - \vartheta_0\big) > m_\varepsilon\right\} \to \mathbf{P}\left\{\xi_{u,+}^* > m_\varepsilon\right\},$$

where $\xi_{u,+}^*$ is the solution of the equation

$$Z\left(\xi_{u,+}^*\right) = \sup_{v>0} Z_u^*(v).$$

Suppose now that the parameter ϑ is a random variable with a given prior density $p(\theta)$, $\vartheta_0 \leq \theta < \beta$. This density is supposed to be continuous and positive. We consider two Bayesian tests: BT1 and BT2.

The BT1 is defined by

$$\widetilde{\phi}_n\big(X^{(n)}\big) = \mathbf{1}_{\left\{\varphi_n^{-1}\left(\widetilde{\vartheta}_n-\vartheta_0\right)>k_\varepsilon\right\}},$$

where k_ε is the solution of the equation

$$\mathbf{P}\left\{\zeta_+^* > k_\varepsilon\right\} = \varepsilon$$

with

$$\zeta_+^* = \frac{\int_0^{+\infty} v\, Z^*(v)\,\mathrm{d}v}{\int_0^{+\infty} Z^*(v)\,\mathrm{d}v}.$$

It belongs to $\mathcal{K}_\varepsilon$ and its power function has the following limit:

$$\beta\big(\widetilde{\phi}_n, u\big) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\varphi_n^{-1}\big(\widetilde{\vartheta}_n - \vartheta_0\big) > k_\varepsilon\right\} \to \mathbf{P}\left\{\zeta_{u,+}^* > k_\varepsilon\right\},$$

where

$$\zeta_{u,+}^{*} = \frac{\int_0^{+\infty} v\, Z_u^*(v)\, \mathrm{d}v}{\int_0^{+\infty} Z_u^*(v)\, \mathrm{d}v}\,.$$

The BT2 is defined by

$$\widetilde{\phi}_n^{\star}\big(X^{(n)}\big) = \mathbf{1}_{\left\{R(X^{(n)})>g_\varepsilon\right\}},$$

where

$$R\big(X^{(n)}\big) = \frac{\varphi_n^{-1}}{p\left(\vartheta_0\right)} \int_{\vartheta_0}^{\beta} \frac{L_n\big(\theta, X^{(n)}\big)}{L_n\big(\vartheta_0, X^{(n)}\big)}\, p(\theta)\, \mathrm{d}\theta$$

and g_ε is the solution of the equation

$$\mathbf{P}\left\{\int_0^{+\infty} Z^*(v)\, \mathrm{d}v > g_\varepsilon\right\} = \varepsilon.$$

It belongs to $\mathcal{K}_\varepsilon$ and its power function has the following limit:

$$\beta\big(\widetilde{\phi}_n^{\star}, u\big) = \mathbf{P}_{\vartheta_u}^{(n)}\left\{\int_0^{+\infty} Z_{n,\vartheta_0}(v)\, \mathrm{d}v > g_\varepsilon\right\} \to \mathbf{P}\left\{\int_0^{+\infty} Z_u^*(v)\, \mathrm{d}v > g_\varepsilon\right\}.$$

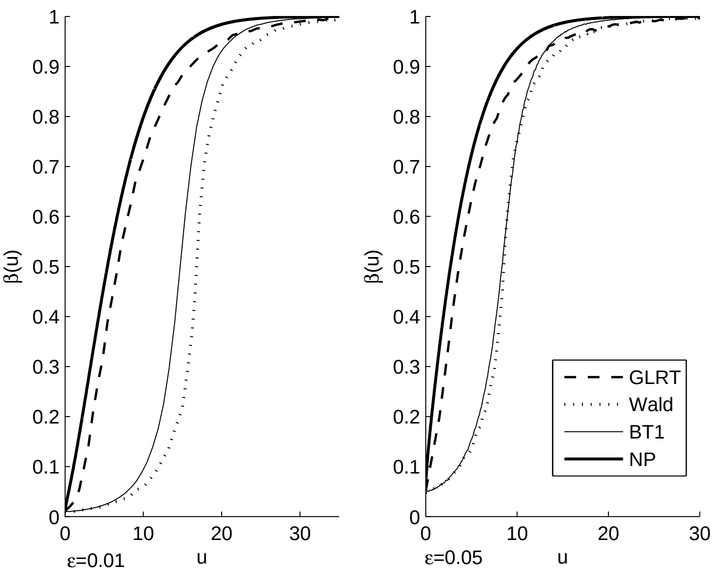

Figure 1. Comparison of limiting power functions for $\varepsilon = 0.01$ and $\varepsilon = 0.05$

The limiting power functions of the GLRT, of the WT and of the BT1 are obtained by means of numerical simulations and are presented in Figure 1 together with the limiting Neyman-Pearson envelope $\beta(u)$,

$u > 0$, given by

$$\beta(u) = 1 - \Phi\big(z_\varepsilon - \sqrt{u}\big),$$

where Φ, as before, is the distribution function of the standard Gaussian low, and z_ε is its quantile of order $1 - \varepsilon$.

We can observe that the limiting power function of the GLRT is the closest to the limiting Neyman-Pearson envelope for small values of u, while the limiting power function of the BT1 is the one that tends to 1 (as u becomes large) the most quickly. We can also see that for $\varepsilon = 0.05$, the limiting power functions of the WT and of the BT1 are close (especially when u is small). Finally, we need to say that all these limiting power functions are perceptibly below the limiting Neyman-Pearson envelope, and that the choice of the asymptotically optimal test remains an open question.

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Дашян С., Янг Л. (Университет Лилля, Лилль, Фучжоуский университет, Фучжоу, Томский государственный университет, Томск, 2018) **Оценивание и проверка гипотезы о малых изменениях интенсивности пуассоновского процесса.**

Аннотация. Рассматривается модель пуассоновских наблюдений, имеющих скачок (точка изменения) в функции интенсивности в случае, когда размер скачка стремится к нулю. Отношение предельного правдоподобия в этом случае сильно отличается от отношения, соответствующего случаю фиксированного размера скачка. Показывается, что отношение предельного правдоподобия является лог-винеровским процессом, и поэтому эта модель асимптотически эквивалентна известной модели сигнала в белом гауссовском шуме. Далее, устанавливаются свойства максимального правдоподобия и байесовских оценок, а также коэффициенты общего правдоподобия, тесты Вальда и два байесовских теста. Приводятся результаты численного моделирования.

Ключевые слова: процесс Пуассона, нерегулярность, точка изменения, малый скачок, статистическое оценивание, проверка гипотез.