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ON AN EXTREMAL PROBLEM FOR NONOVERLAPPING DOMAINS[1]

The paper considers the problem of finding the range of the functional $I = J(f(z_0), \overline{f(z_0)}, F(\zeta_0), \overline{F(\zeta_0)})$ defined on the class $\mathfrak{M}$ of functions pairs $(f(z), F(\zeta))$ that are univalent in the system of the disk and the interior of the disk, using the method of internal variations. We establish that the range of this functional is bounded by the curve whose equation is written in terms of elliptic integrals, depending on the parameters of the functional *I*.

Keywords: *Method of internal variations, Univalent function, Nonoverlapping domains, Functional range, Elliptic integrals.*

1. Introduction

In the geometric theory of univalent functions there are a number of papers and books devoted to the problem of nonoverlapping domains. These problems were developed by M.A. Lavrentiev [1], G.M. Goluzin [2], James A. Jenkins [3], Z. Nehari [4], N.A. Lebedev [5], M. Shiffer [6], R. Kühnau [7] and others. A summary of results in this area is contained in [8].

Let D and D^* be nonoverlapping simply connected domains in the w-plane such that $0 \in D$ and $\infty \in D^*$. Assume that $f: E \to D$ and $F: E^* \to D^*$ are holomorphic and meromorphic (respectively) univalent functions normalized by the conditions $f(0) = 0$ and $F(\infty) = \infty$. Here $E = \{z \in \mathbb{C} : |z| < 1\}$ and $E^* = \{\zeta \in \mathbb{C} : |z| > 1\}$. The family of all such pairs $(f(z), F(\zeta))$ is called the class $\mathfrak{M}$. Some extremal problems for this class were studied in [5, 9, 10].

Let $J: G \to \mathbb{C}$ and $J = J(\omega_1, \omega_2, \omega_3, \omega_4)$ is an analytic in some domain $G \subset \mathbb{C}^4$ nonhomogeneous and nonconstant function.

Now we fix an arbitrary points $z_0 \in E$, $\zeta_0 \in E^*$ and define on the class $\mathfrak{M}$ a functional

$$I: \mathfrak{M} \to \mathbb{C}, \ I(f, F) = J\left(f(z_0), \overline{f(z_0)}, F(\zeta_0), \overline{F(\zeta_0)}\right). \quad (1)$$

This paper considers the problem of finding the range Δ of the functional I on the class $\mathfrak{M}$. Since together with pair $(f(z), F(\zeta))$ the class $\mathfrak{M}$ contains the following pairs of functions $\left(f\left(ze^{i\varphi}\right), F\left(\zeta e^{i\psi}\right)\right)$ for any parameters $\varphi, \psi \in \mathbb{R}$, then we reduce the initial problem to an equivalent one for the functional

$$I = J\left(f(r), \overline{f(r)}, F(\rho), \overline{F(\rho)}\right),$$

where $r = |z_0| \in (0,1)$, $\rho = |\zeta_0| \in (1, +\infty)$. Further on we will solve this problem. We note that the class of such problems has a long history. Its special cases have been studied in the works [11, 12].

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Taking into account that the class $\mathfrak{M}$ contains the pair of functions $\left(f(zt), F\left(\zeta t^{-1}\right)\right)$ for any $t \in [0,1]$ (see [9]), we have that the range Δ of the functional (1) is a connected set. Note that if the class $\mathfrak{M}$ complement the pair of functions $(f(z), \infty)$, where $f(z)$, $f(0)=0$, is a holomorphic univalent function in E, then Δ will be a closed set [13]. Hence, it suffices to find the boundary Γ of the set Δ.

A point $I_0 \in \Gamma$ is called a *nonsingular boundary point* if there exists an exterior point I_e for Δ such that the distance between I_e and I_0 is equal to the distance between I_e and the set Δ, i.e.

$$|I_0 - I_e| = \inf_{I \in \Delta} |I - I_e|. \tag{2}$$

The set Γ_0 of nonsingular boundary points is dense in Γ [14]. Thus, the initial problem of finding the range Δ of (1) is replaced by an equivalent extremal problem: find the minimum of the real-valued functional $|I - I_e|$ on $\mathfrak{M}$ for all possible $I_e \notin \Delta$.

The functions giving nonsingular boundary points of a functional are called the *boundary functions* of this functional; i.e., these are the functions at which the values of the functional are nonsingular boundary points.

In this paper, to solve the problem we apply Schiffer's method of internal variations [15] using pairs of varied functions from [5]. In [10, 16] this method has been developed to studying the range of some functionals defined on the classes of pairs functions.

Main results are given in the following Section 2.

2. Differential equations for the boundary functions. The equation of the boundary of Δ

Now write the variational formulas for boundary functions $f(z)$ and $F(\zeta)$ on the class $\mathfrak{M}$ in the form

$$f_\varepsilon(z) = f(z) + \varepsilon P(z) + o(z, \varepsilon),$$

$$F_\varepsilon(\zeta) = F(\zeta) + \varepsilon Q(\zeta) + o(\zeta, \varepsilon)$$

for ε positive and sufficiently small. Because the functional (1) is Gâteaux differentiable, we can rewrite it in the form

$$I_* = I + \varepsilon \left\{ \frac{\partial J(\omega_0)}{\partial \omega_1} P(r) + \frac{\partial J(\omega_0)}{\partial \omega_2} \overline{P(r)} + \frac{\partial J(\omega_0)}{\partial \omega_3} Q(\rho) + \frac{\partial J(\omega_0)}{\partial \omega_4} \overline{Q(\rho)} \right\} + o(\varepsilon),$$

where $I_* = I\left(f_\varepsilon(z), F_\varepsilon(\zeta)\right)$, $\omega_0 = \left(f(r), \overline{f(r)}, F(\rho), \overline{F(\rho)}\right) \in G$.

Let $(f(z), F(\zeta))$ be a boundary pair of functions of this functional giving the point I_0. The equality (2) implies that

$$|I_* - I_e| \geq |I_0 - I_e|.$$

From here each boundary pair of functions satisfies the necessary condition

$$\operatorname{Re}\left[pP(r) + qQ(\rho)\right] \geq 0 \tag{3}$$

where $$p = e^{-i\alpha} \frac{\partial J(\omega_0)}{\partial \omega_1} + e^{i\alpha} \overline{\left(\frac{\partial J(\omega_0)}{\partial \omega_2}\right)}, \quad q = e^{-i\alpha} \frac{\partial J(\omega_0)}{\partial \omega_3} + e^{i\alpha} \overline{\left(\frac{\partial J(\omega_0)}{\partial \omega_4}\right)}$$

and $$\alpha = \arg\left(I - I_\varepsilon\right).$$

Lemma 1. Let $f(z), F(\zeta)$ be boundary functions of functional (1). Then the union of domains D and D^* has no exterior points in $\overline{\mathbb{C}}_w$.

Proof. Suppose that $D \cup D^*$ has at least one exterior point w_0 in the w-plane. Then by definition of the exterior point there exists a neighborhood of the point w_0 consisting of exterior points. Using the pair of varied functions

$$f(z,\varepsilon)=f(z)+\varepsilon A_0\frac{f(z)}{f(z)-w_0},\ F(\zeta,\varepsilon)=F(\zeta)+\varepsilon A_0\frac{F(\zeta)}{F(\zeta)-w_0},$$

where w_0 is an exterior point for D and D^* simultaneously and A_0 is an arbitrary complex constant, as the comparison pair in (3), rewrite it as

$$\operatorname{Re}\left(A_0\frac{R(w_0)}{(f(r)-w_0)(F(\rho)-w_0)}\right)\ge 0, \tag{4}$$

where $R(w_0)$ is a linear polynomial. The fraction in the condition (4) is equal to zero, otherwise in view of the arbitrariness of $\arg A_0$ we can choose it so that the left side will be negative. Therefore $R(w_0)=0$. It is possible only for a single point, while inequality (4) should be performed for any point from the neighborhood of the w_0. This contradiction proves the lemma.

Further, to obtain differential equations for the boundary functions of the functional (1) we consider the following pairs of variational formulas from [5]:

1)
$$f(z,\varepsilon)=f(z)+\varepsilon A_0\left(\frac{f(z)}{f(z)-f(z_0)}-\frac{f(z_0)}{z_0 f'^2(z_0)}\frac{zf'(z)}{z-z_0}\right)+$$
$$+\varepsilon\overline{A_0}\,\overline{\frac{f(z_0)}{z_0 f'^2(z_0)}}\frac{z^2 f'(z)}{1-\overline{z_0}z}+o(z,\varepsilon), \tag{5}$$
$$F(\zeta,\varepsilon)=F(\zeta)+\varepsilon A_0\frac{f(z)}{f(z)-F(\zeta_0)},$$

where $z_0\in E$, A_0 is an arbitrary complex constant;

2)
$$f(z,\varepsilon)=f(z)+\varepsilon A_0\frac{f(z)}{f(z)-F(\zeta_0)},$$
$$F(\zeta,\varepsilon)=F(\zeta)+\varepsilon A_0\left(\frac{F(\zeta)}{F(\zeta)-F(\zeta_0)}-\frac{F(\zeta_0)}{\zeta_0^2 F'^2(\zeta_0)}\frac{\zeta^2 F'(\zeta)}{\zeta-\zeta_0}\right)+$$
$$+\varepsilon\overline{A_0}\,\overline{\frac{F(\zeta_0)}{\zeta_0^2 F'^2(\zeta_0)}}\frac{\zeta F'(\zeta)}{1-\overline{\zeta_0}\zeta}+o(\zeta,\varepsilon), \tag{6}$$

where $\zeta_0\in E^*$, A_0 is an arbitrary complex constant.

Theorem 2. Every boundary pair of functions $(f(z),F(\zeta))$ of the functional (1) satisfies in E and E^* the system of functional-differential equations

$$\frac{(C_1 f(z)-C_2)(f'(z))^2}{f(z)(f(z)-f(r))(f(z)-F(\rho))}=\frac{A}{z(r-z)(1-rz)}, \tag{7}$$

$$\frac{(C_1F(\zeta)-C_2)(F'(\zeta))^2}{F(\zeta)(F(\zeta)-f(r))(F(\zeta)-F(\rho))}=\frac{B}{\zeta(\rho-\zeta)(1-\rho\zeta)}, \tag{8}$$

where $C_1 = pf(r)+qF(q)$, $C_2=(p+q)f(r)F(q)$,

$$A=-(1-r^2)rp\,f'(r)>0,\ B=(\rho^2-1)\rho\,qF'(\rho)>0.$$

Proof. If we choose the variational formula (5), then (3) takes the form

$$\operatorname{Re}\left[pA_0\frac{f(r)}{f(r)-f(z_0)}-pA_0\frac{rf'(r)}{r-z_0}\frac{f(z_0)}{z_0f'^2(z_0)}+p\overline{A_0}\frac{\overline{r^2f'(r)}}{1-r\overline{z_0}}\overline{\frac{f(z_0)}{z_0f'^2(z_0)}}+qA_0\frac{F(\rho)}{F(\rho)-f(z_0)}\right]\geq 0.$$

Replacing the third summand under the real part by its conjugate, we have

$$\operatorname{Re}A_0\left[\frac{pf(r)}{f(r)-f(z_0)}-p\frac{rf'(r)}{r-z_0}\frac{f(z_0)}{z_0f'^2(z_0)}+p\frac{\overline{r^2f'(r)}}{1-r\overline{z_0}}\frac{f(z_0)}{z_0f'^2(z_0)}+\frac{qF(\rho)}{F(\rho)-f(z_0)}\right]\geq 0.$$

In this condition, the expression in parentheses is equal to zero; otherwise, under an appropriate choice of $\arg A_0$ we would get that the left-hand side of the last inequality is negative. This leads to the equality

$$\frac{pf(r)}{f(r)-f(z_0)}+\frac{qF(\rho)}{F(\rho)-f(z_0)}=\frac{f(z_0)}{z_0f'^2(z_0)}\left(\frac{rpf'(r)}{r-z_0}-\overline{pf'(r)}\frac{r^2}{1-rz_0}\right).$$

Since, in this equation, z_0 is an arbitrary point of E, replacing z_0 by z, and in view of $pf'(r)<0$, we obtain a differential equation for the boundary function $f(z)$. The calculations show that it has the form (7).

The deduction of (8) repeats (7); for this we must apply (3) together with the variational formulas (6) and use inequality $qF'(\rho)>0$. The theorem is proved.

From the analytic theory of differential equations [17], we conclude that the boundary functions $f(z)$ and $F(\zeta)$ satisfying their equations are holomorphic not only in E and E^*, but also on the unit circle $|z|=|\zeta|=1$. From here and because the union $D\cup D^*$ does not contain exterior points, we have that the domains D and D^* are bounded by some closed analytic Jordan curve.

Further, to find the equation of the boundary of the range Δ of the functional (1) we integrate (7) and (8).

Extract the square root from both sides of (7) and integrate the result by z from 0 to r. Consider the left-hand side:

$$J=\int_0^r\sqrt{\frac{C_1f(z)-C_2}{f(z)(f(z)-f(r))(f(z)-F(\rho))}}f'(z)dz.$$

Changing the integration variable $t=f(z)/f(r)$, we have

$$J=a\int_0^1\frac{t-b}{\sqrt{t(1-t)(1-ct)(t-b)}}dt,$$

where $a=\sqrt{C_1\frac{f(r)}{F(\rho)}}$, $b=\frac{C_2}{C_1f(r)}$, $c=\frac{f(r)}{F(\rho)}$.

Putting $t=1/x$ in J, we infer

$$J = a\int_1^{\infty}\frac{dx}{x\sqrt{(x-1)(x-c)(1-bx)}} - ab\int_1^{\infty}\frac{dx}{\sqrt{(x-1)(x-c)(1-bx)}}.$$

Performing the change of variables $y=b(x-1)/(bx-1)$ in the integrals, after transformations we obtain

$$J=-ph\,\mathbf{\Pi}(n,k),$$

where $$\mathbf{\Pi}(n,k)=\int_0^{\pi/2}\frac{dt}{(1+n\sin^2 t)\sqrt{1-k^2\sin^2 t}}$$

is the complete elliptic integral of the third kind,

$$h=\sqrt{\frac{(f(r)-F(\rho))f^2(r)}{C_2}},\; n=-\frac{1}{b},\; k=\sqrt{\frac{q}{p+q}}.$$

Here $\sqrt{1-k^2\sin^2 t}$ stands for the branch of the function assuming 1 at $t\to 0$.

Now integrate the right-hand side:

$$J=\sqrt{A}\int_0^{r}\frac{dz}{\sqrt{z(r-z)(1-rz)}}.$$

Changing the integration variable $x=z/r$, we have

$$J=2\sqrt{A}\,\mathbf{K}(r),$$

where $$\mathbf{K}(r)=\int_0^{\pi}\frac{dt}{\sqrt{1-r^2\sin^2 t}}$$

is the complete elliptic integral of the first kind.

Thus, upon integration, we can rewrite (7) as

$$-ph\,\mathbf{\Pi}(n,k)=\sqrt{A}\,\mathbf{K}(r). \quad (9)$$

Integrate (8) after extracting the square root with respect to ζ from ρ to ∞. First consider the left-hand side:

$$L=a^*\int_{\rho}^{\infty}\sqrt{\frac{C_1F(\zeta)-C_2}{F(\zeta)(F(\zeta)-F(\rho))\big(F(\zeta)-f(r)\big)}}F'(\zeta)d\zeta.$$

Changing the integration variable $t=F(\zeta)/F(\rho)$, we have

$$L=a^*\int_1^{\infty}\frac{t-b^*}{\sqrt{t(1-t)(1-c^*t)(t-b^*)}}dt,$$

where $$a^*=\sqrt{C_1\frac{F(\rho)}{f(r)}},\; b^*=\frac{C_2}{C_1F(\rho)},\; c^*=\frac{F(\rho)}{f(r)}.$$

Putting $t=1/x$ in L, we infer

$$L=a^*\int_0^{1}\frac{dx}{x\sqrt{(x-1)(x-c^*)(1-b^*x)}}-a^*b^*\int_0^{1}\frac{dx}{\sqrt{(x-1)(x-c^*)(1-b^*x)}}.$$

Performing the change of variables $u = b^*(1-x)/(b^*-1)$ in the integrals, after calculation we come to equality

$$L = 2\left(l\mathbf{\Pi}(\phi, m, k) - h_0\mathbf{F}(\phi, k)\right),$$

where
$$\mathbf{F}(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$$

is the incomplete elliptic integral of the first kind;

$$\mathbf{\Pi}(\phi, m, k) = \int_0^{\phi} \frac{dt}{(1+m\sin^2 t)\sqrt{1-k^2\sin^2 t}}$$

is the incomplete elliptic integral of the third kind;

$$l = C_1\sqrt{\frac{c^*}{(p+q)\left(f(r)-F(\rho)\right)}},$$

$$\phi = \arcsin\frac{1}{k\left(1-c^*\right)},\ m = k\left(c^*-1\right),\ h_0 = \frac{f(r)}{h}.$$

On the right-hand side

$$J = \sqrt{B}\int_{\rho}^{\infty} \frac{d\zeta}{\sqrt{\zeta(\rho-\zeta)(1-\rho\zeta)}},$$

changing the integration variable $\tau = \rho/\zeta$, we have

$$L = 2\frac{\sqrt{B}}{\rho}\mathbf{K}\left(\frac{1}{\rho}\right).$$

Hence, integrating (8), we obtain

$$\left(l\ \mathbf{\Pi}(\phi, m, k) - h_0\mathbf{F}(\phi, k)\right) = \frac{\sqrt{B}}{\rho}\mathbf{K}\left(\frac{1}{\rho}\right). \tag{10}$$

Now in the w-plane, take the point $F(1) = f(-e^{i\alpha})$. Integrate the equalities that are obtained from the system (7)–(8) by extracting the square root in first over z from 0 to -1, and then over the arc $|z| = 1$ counterclockwise from -1 to $-e^{i\alpha}$, and in the second, over ζ from 1 to ρ.

Write the integrals on the left-hand side of (7):

$$\int_0^{f(-1)} \sqrt{\frac{C_1 w - C_2}{w\left(w-F(\rho)\right)\left(w-f(r)\right)}}dw,$$

$$\int_{f(-1)}^{F(1)} \sqrt{\frac{C_1 w - C_2}{w\left(w-F(\rho)\right)\left(w-f(r)\right)}}dw.$$

Proceed with the integral on the left-hand side of (8)

$$\int_{F(1)}^{F(\rho)} \sqrt{\frac{C_1 w - C_2}{w\left(w-F(\rho)\right)\left(w-f(r)\right)}}dw.$$

Summing up these integrals, we have

$$T = \int_0^{F(\rho)} \sqrt{\frac{C_1 w - C_2}{w\left(w - F(\rho)\right)\left(w - f(r)\right)}}\, dw .$$

Making the change of variables $t = w / F(\rho)$, note that

$$T = a^* \int_0^1 \frac{t - b^*}{\sqrt{t(1-t)(1-c^* t)(t-b^*)}}\, dt .$$

Using $t = 1/x$, we find

$$T = a^* \int_1^\infty \frac{dx}{x\sqrt{(x-1)(x-c^*)(1-b^* x)}} - a^* b^* \int_1^\infty \frac{dx}{\sqrt{(x-1)(x-c^*)(1-b^* x)}} .$$

Changing the integration variable $y = b^*(1-x)/(b^* x - 1)$ in the integrals and performing the corresponding transformations, we obtain

$$T = 2iqh^* \mathbf{\Pi}(n^*, k') ,$$

where $$h^* = \sqrt{\frac{(f(r) - F(\rho))F^2(\rho)}{C_2}} ,\ k' = \sqrt{1-k^2} ,\ n^* = -\frac{1}{b^*} .$$

Integrate the right-hand sides of (7). First, integrate over z from 0 to -1

$$T_1 = \sqrt{A} \int_{-1}^0 \frac{dz}{\sqrt{z(r-z)(1-rz)}} .$$

Performing the change of the integration variable $x = (1+z)/(1-z)$ and applying a formula from [18], we obtain

$$T_1 = \sqrt{A} i\ \mathbf{K}(\sqrt{1-r^2}) .$$

Integrate the right-hand side over the arc β of the unit circle counterclockwise from -1 to $-e^{i\alpha}$:

$$T_2 = \sqrt{A} \int_\beta \frac{dz}{\sqrt{z(r-z)(1-rz)}} .$$

Substituting $z = -e^{i\varphi}$ in T_2, where $0 \le \varphi \le \alpha$, we infer

$$T_2 = \sqrt{A} \int_0^\alpha \frac{d\left(-e^{i\varphi}\right)}{\sqrt{-e^{i\varphi}(r + e^{i\varphi})(1 + re^{i\varphi})}} .$$

Performing the change of variable $t = -e^{i\varphi}$, we find

$$T_2 = -\frac{2\alpha}{\pi} \sqrt{A}\ \mathbf{K}(r) .$$

Finally, integrate the right-hand side of (8) over ζ from 1 to ρ:

$$T_3 = \sqrt{B} \int_1^\rho \frac{d\zeta}{\sqrt{\zeta(\rho-\zeta)(1-\rho\zeta)}} .$$

Changing the integration variable $u = \rho / \zeta$, we have

$$T_3 = \frac{\sqrt{B}}{\rho}\,\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right).$$

Summing T_1, T_2 and T_3 yields

$$\sqrt{A}i\,\mathbf{K}(\sqrt{1-r^2}) + \frac{\sqrt{B}}{\rho}\,\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right) - \frac{2\alpha}{\pi}\sqrt{A}\,\mathbf{K}(r).$$

Thus, integrating the equalities (7) and (8), we come to

$$qh^*\mathbf{\Pi}(n^*,k') = \frac{\sqrt{A}}{2}\,\mathbf{K}(\sqrt{1-r^2}) + \frac{\sqrt{B}}{2\rho}\,\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right) - \frac{\alpha}{\pi}i\sqrt{A}\,\mathbf{K}(r).$$

Excluding the constants $\sqrt{A}$ and $\sqrt{B}$ from this equality by using (9) and (10), we obtain the following equation

$$-\frac{qh^*}{ph}\frac{\mathbf{\Pi}(n^*,k')}{\mathbf{\Pi}(n,k)} = \frac{1}{2}\frac{\mathbf{K}(\sqrt{1-r^2})}{\mathbf{K}(r)} - \frac{1}{2}\frac{l\,\mathbf{\Pi}(\phi,m,k) - h_0\mathbf{F}(\phi,k)}{ph\mathbf{\Pi}(n,k)}\frac{\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right)}{\mathbf{K}\left(\frac{1}{\rho}\right)} + \frac{\alpha}{\pi}i. \quad (11)$$

Hence, we have proved:

Theorem 3. The range Δ of the functional (1) on the class $\mathfrak{M}$ is bounded by the curve defined by equation (11) for $0 \le \alpha < 2\pi$.

Note that Theorem 3 implies the following result, established in [10].

Corollary 4. Let $(f(z), F(\zeta)) \in \mathfrak{M}$ and r, ρ are fixed points in E and E^* respectively. Then the range of the functional $\xi = \frac{1}{2}\ln\frac{f(r)}{F(\rho)}$ is bounded by the curve

$$\xi(\lambda) = \frac{1}{2}\ln\left(-16d\prod_{n=1}^{\infty}\left(\frac{1+d^{2n}}{1-d^{2n-1}}\right)^8\right), \quad (12)$$

where $d = e^{-\pi\mu}$, $\mu = \frac{1}{2}\left\{\frac{\mathbf{K}(\sqrt{1-r^2})}{\mathbf{K}(r)} + \frac{\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right)}{\mathbf{K}\left(\frac{1}{\rho}\right)}\right\} + \lambda i$, $0 \le \lambda < 2$.

In Figures 1 and 2 below we can see the curve (12) for some fixed parameters r and ρ.

Corollary 4 allow us to obtain the new estimates for the moduli of the functionals on the class $\mathfrak{M}$.

Proposition 5. On the class $\mathfrak{M}$ for any $r \in (0,1)$ and $\rho \in (1,\infty)$, the following inequality holds

$$\left|\frac{f(r)}{F(\rho)}\right| \le e^{2\xi(0)}.$$

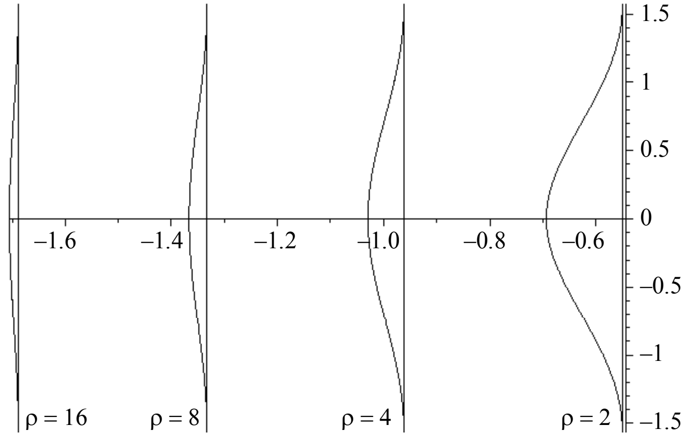

Fig. 1. Curve (12) for $r = 0.5$ and $\rho = 2, 4, 8, 16$

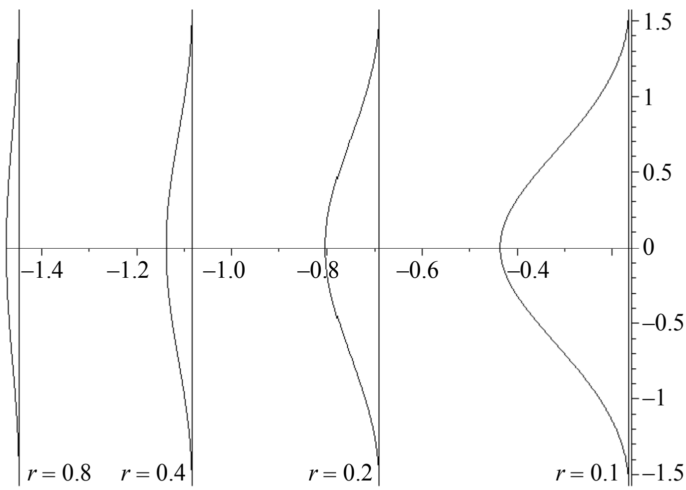

Fig. 2. Curve (12) for $r = 0.1,\ 0.2,\ 0.4,\ 0.8$ and $\rho = 2$

Proof. Using the equality (12) and formulas from [18], it is easy to show that

$$\left|\frac{f(r)}{F(\rho)}\right| \le 16e^{-\pi\nu}\cdot\left|e^{-\pi\lambda i}\right|\cdot\prod_{n=1}^{\infty}\left(\frac{\left|1+\left(e^{-\pi\nu}\cdot e^{-\pi\lambda i}\right)\right|^{2n}}{\left|1-\left(e^{-\pi\nu}\cdot e^{-\pi\lambda i}\right)\right|^{2n-1}}\right)^{8} \le$$

$$\le 16e^{-\pi\nu}\cdot\prod_{n=1}^{\infty}\left(\frac{\left|1+\left(e^{-\pi\nu}\cdot\left|e^{-\pi\lambda i}\right|\right)\right|^{2n}}{\left|1-\left(e^{-\pi\nu}\cdot\left|e^{-\pi\lambda i}\right|\right)\right|^{2n-1}}\right)^{8} = e^{2\xi(0)},$$

where
$$\nu = \frac{1}{2}\left\{\frac{\mathbf{K}(\sqrt{1-r^2})}{\mathbf{K}(r)} + \frac{\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right)}{\mathbf{K}\left(\frac{1}{\rho}\right)}\right\}.$$

Hence Proposition 5. Next we establish the auxiliary inequality.

Proposition 6. Let $\xi(\lambda)$ describes by the equality (12). Then

$$e^{2\xi(0)} \le \frac{r}{\sqrt{(1-r^2)(\rho^2-1)}}$$

for any $r \in (0,1)$ and $\rho \in (1,\infty)$.

Proof. Taking into account the formulas from [18] for the elliptic integrals, we have

$$\frac{r}{\sqrt{1-r^2}} = 4\sqrt{q_1}\prod_{n=1}^{\infty}\left(\frac{1+q_1^{\,2n}}{1+q_1^{\,2n-1}}\right)^4, \quad q_1 = e^{-\pi\frac{\mathbf{K}(\sqrt{1-r^2})}{\mathbf{K}(r)}}.$$

$$\frac{1}{\sqrt{\rho^2-1}} = 4\sqrt{q_2}\prod_{n=1}^{\infty}\left(\frac{1+q_2^{\,2n}}{1+q_2^{\,2n-1}}\right)^4, \quad q_2 = e^{-\pi\frac{\mathbf{K}(\sqrt{1-1/\rho^2})}{\mathbf{K}(1/\rho)}}.$$

One can check that for any $m \in \mathbb{N}$

$$\left(1+e^{a+b}\right)^m \le \left(1+e^a\right)^{m/2}\left(1+e^b\right)^{m/2}, \quad \left(1-e^{a+b}\right)^m \ge \left(1-e^a\right)^{m/2}\left(1-e^b\right)^{m/2}.$$

From here we obtain

$$e^{2\xi(0)} = 16e^{-\pi\nu}\cdot\prod_{n=1}^{\infty}\left(\frac{1+e^{-2n\pi\nu}}{1+e^{-(2n-1)\pi\nu}}\right)^8 \le$$

$$\le 4e^{-\pi a}\prod_{n=1}^{\infty}\left(\frac{1+e^{-2n\pi a}}{1-e^{-(2n-1)\pi a}}\right)^4 \times 4e^{-\pi b}\prod_{n=1}^{\infty}\left(\frac{1+e^{-2n\pi b}}{1-e^{-(2n-1)\pi b}}\right)^4 = \frac{r}{\sqrt{(1-r^2)(\rho^2-1)}},$$

where $$\nu = a+b, \; a = \frac{1}{2}\frac{\mathbf{K}(\sqrt{1-r^2})}{\mathbf{K}(r)}, \; b = \frac{1}{2}\frac{\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right)}{\mathbf{K}\left(\frac{1}{\rho}\right)}.$$

Hence Proposition 6.

From Proposition 5 and Proposition 6 we come to the corollaries.

Corollary 7. On the class $\mathfrak{M}$ for any $r \in (0,1)$ and $\rho \in (1,\infty)$, the following inequality holds

$$\left|\frac{f(r)}{F(\rho)}\right| \le \frac{r}{\sqrt{(1-r^2)(\rho^2-1)}}.$$

Corollary 8. On the class $\mathfrak{M}$ the following estimates hold:

1) the range of functional $\frac{f'(0)}{F(\rho)}$ is a punctured disc

$$\left|\frac{f'(0)}{F(\rho)}\right| \le 4\exp\left\{-\frac{\pi}{2}\frac{\mathbf{K}\left(\sqrt{1-\frac{1}{\rho^2}}\right)}{\mathbf{K}\left(\frac{1}{\rho}\right)}\right\},$$

2) the range of functional $\frac{f(r)}{F'(\infty)}$ is a punctured disc

$$\left|\frac{f(r)}{F'(\infty)}\right| \le 4\exp\left\{-\frac{\pi}{2}\frac{\mathbf{K}(\sqrt{1-r^2})}{\mathbf{K}(r)}\right\}.$$

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В статье методом внутренних вариаций решается задача о нахождении множества значений функционала $I = J\left(f(z_0), \overline{f(z_0)}, F(\zeta_0), \overline{F(\zeta_0)}\right)$, определенного на классе $\mathfrak{M}$ пар функций $(f(z), F(\zeta))$ однолистных в системе круг – внешность круга. Устанавливается, что множество значений функционала ограничено кривой, уравнение которой записано через эллиптические интегралы, зависящие от параметров функционала I.

Ключевые слова: метод внутренних вариаций, однолистные функции, неналегающие области, множество значений функционала, эллиптические интегралы.

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