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Method for testing robust control quality for linear one-dimensional discrete control systems

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Abstract. The article deals with one-dimensional linear control objects in discrete time. It is assumed that the operators of the control object have structural and parametric inaccuracies in the description. In this paper, a criterion for the robust quality of control is obtained for a system consisting of a discrete control object with structural perturbations and a modal controller. Based on this criterion, a numerical method for checking the robust quality of control has been developed. The effectiveness of the method is illustrated by an example.

Keywords: structural perturbations; modal controller; robust control quality.

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Метод проверки робастного качества управления для линейных одномерных дискретных систем управления

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Аннотация. Рассматриваются одномерные линейные объекты управления в дискретном времени. Предполагается, что операторы объекта управления имеют структурно-параметрические неточности в описании. Получен критерий робастного качества управления для системы, состоящей из дискретного объекта управления со структурными возмущениями и модального регулятора. На основе данного критерия разработан численный метод проверки робастного качества управления. Эффективность метода проиллюстрирована примером.

Ключевые слова: структурные возмущения; модальный регулятор; робастное качество управления.

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Introduction

Technological processes with different tempo components are widespread [1, 2]. A characteristic feature of the models of such processes is the allocation in the model of operators of the so-called "basic dynamics", which describes that part of the control object that is subject to regulation, and operators of "structural disturbances" – they include those parts of the control object that already have the properties of stability and a given quality management [3. pp. 29–30]. When synthesizing a controller, structural perturbations, as a rule, are not taken into account [4–5], as a result, uncertainty arises in the transfer function (TF) of a closed system. Since the properties of stability and quality of control of the system are determined by the location of the poles of its PF, the question arises: under what operators of structural perturbations in the control object will the closed system still retain the properties of stability (robust stability) and quality of control (robust quality of control)?

In the modal control scheme [3. pp. 8–21, 6. pp. 5–20] the quality of control is given as an area S on the complex plane; the S area determines the desired location of the PF poles. Consequently, the issues of studying (testing) robust stability and robust quality of control can be considered from a unified standpoint: is it required to check whether the zeros of a given family of polynomials belong to the area S?

The problem of studying robust stability and robust quality of control is widely presented in the literature. There are three main directions in which this problem is solved: 1) the principle of "zero exclusion" [7–12]; 2) H^∞ theory [13–14]; 3) LMI method [15–17]. However, the general formulation of the criterion of robust control quality (independent of the shape of the area S) has not yet been obtained [11. p. 227].

In this article, in development of the results obtained in [12], we develop a method for testing the robust quality of control.

The following notation is adopted:

$\doteq$ is equal by definition;

R^n is the n-dimensional space of real numbers;

C^n is the n-dimensional space of complex numbers;

C^1 is the complex plane; $^{\mathrm{T}}$ is the transposition operation;

E is the identity matrix of the corresponding dimension;

0 is a column vector consisting entirely of zeros;

j is the imaginary unit; s is a variable (generally complex);

s^* is the number complex conjugate of s;

t is discrete time ($t = 0, \pm 1, \pm 2, \ldots$);

z is advance operator by one cycle: $zx(t) = x(t+1)$;

S is the area on C^1;

∂S is the boundary of the area S;

int S is the interior of the area S.

Let $f(\mathbf{x})$ be a function of the vector argument $\mathbf{x}$ defined on the area $X \subset R^n$; denote

$$|f|_+ \doteq \max_{\mathbf{x}\in X} |f(\mathbf{x})|, \quad |f|_- \doteq \min_{\mathbf{x}\in X} |f(\mathbf{x})|. \tag{1}$$

View operator

$$f(n,z) = \sum_{i=0}^{n} f_i \cdot z^i, \tag{2}$$

is called a *polynomial operator* of degree n. Replacing z with s in (2), where s is a variable, we obtain an algebraic polynomial

$$f(n,s) = \sum_{i=0}^{n} f_i \cdot s^i.$$

The set of zeros of the polynomial $f(n, s)$ is denoted by $\Lambda(f)$:

$$\Lambda(f) \doteq \left\{ \lambda_i \in C^1 : f(n,\lambda_i) = 0, \quad i \in \overline{1,n} \right\}.$$

Family of operators of the form

$$f(n,F,z)=\left\{f(n,\mathbf{f},z)=\sum_{i=0}^{n} f_i\cdot z^i:\quad \mathbf{f}\doteq(f_0,\quad\cdots,\quad f_n)\in F\right\}, \tag{3}$$

$$F=\left\{\mathbf{f}\in R^{n+1}: f_i\in\left[f_i^0-\Delta f_i;\quad f_i^0+\Delta f_i\right],\right.$$

$$\left. f_n^0\neq 0,\quad \Delta f_i\geq 0,\quad i=\overline{0,n}\right\},$$

is called an *interval polynomial operator* of degree n.

Replacing z with s in (3), we obtain an *interval polynomial*:

$$f(n,F,s)=\left\{f(n,\mathbf{f},s)=\sum_{i=0}^{n} f_i\cdot s^i:\quad \mathbf{f}\in F\right\}. \tag{4}$$

Interval polynomial (4) can be represented as

$$f(n,F,s)=f^0(n,\mathbf{f}^0,s)+\Delta f(n,\Delta F,s), \tag{5}$$

where

$$f^0(n,\mathbf{f}^0,s)=\sum_{i=0}^{n} f_i^0\cdot s^i,\ \mathbf{f}^0\doteq\left(f_0^0,\quad\cdots,\quad f_n^0\right),$$

ordinary polynomial, and

$$\Delta f(n,\Delta F,s)=\left\{f(n,\delta\mathbf{f},s)=\sum_{i=0}^{n}\delta f_i\cdot s^i:\delta\mathbf{f}\doteq(\delta f_0,...,\delta f_n)\in\Delta F\right\},$$

$$\Delta F=\left\{\delta\mathbf{f}\in R^{n+1}:\quad \delta f_i\in[-\Delta f_i;\quad \Delta f_i],\quad i=\overline{0,n}\right\},$$

interval polynomial with symmetric coefficient uncertainty intervals.

Let **A** be a matrix whose coefficients are complex numbers, then we denote by $\mathbf{A}_{\mathrm{Re}}$ and $\mathbf{A}_{\mathrm{Im}}$ the matrices whose coefficients are, respectively, the real and imaginary parts of the coefficients of the matrix **A**, in this sense we denote

$$\mathbf{A}_{\mathrm{Re}}\doteq\mathrm{Re}(\mathbf{A}),\quad \mathbf{A}_{\mathrm{Im}}\doteq\mathrm{Im}(\mathbf{A}).$$

1. Synthesis of a modal controller for linear control objects with structural perturbations and statement of the problem of testing the robust quality of control

Let a linear one-dimensional discrete control object be given by a model of the form:

$$v(l,V,z)\cdot a(n,A,z)y(t)=w(h,W,z)\cdot b(m,B,z)u(t),\quad n>m,\quad l\geq h, \tag{6}$$

here u is the input (control) signal, y is the output (controlled) signal, $a(n, A, z)$, $b(m, B, z)$, $v(l, V, z)$, $w(h, W, z)$ are interval operators of the form (5) such that

$$a_n^0=1,\quad \Delta a_n=0,\quad v_0^0=1,\quad \Delta v_0=0,\quad w_0^0=1,\quad \Delta w_0=0.$$

Model

$$a^0(n,\mathbf{a}^0,z)y(t)=b^0(m,\mathbf{b}^0,z)u(t), \tag{7}$$

belonging to the family of models (6), we will call "nominal"; interval operators $a(n, A, z)$, $b(m, B, z)$ call "basic dynamics"; operators $v(l, V, z)$, $w(h, W, z)$ call "structural perturbations".

The quality of control will be assigned in the form of an area S, which determines the admissible location of the PF poles on C^1. We assume that the area S satisfies the following conditions: it is located inside a circle of unit radius; simply connected; also holds $s^*\in S$ for any point $s\in S$.

In addition, based on the meaning of the problem, we require that the operators of structural perturbations $v(l, V, z)$, $w(h, W, z)$ satisfy the expressions:

$$\Lambda(v)\subset\mathrm{int}\,S,\quad \Lambda(w)\subset\mathrm{int}\,S. \tag{8}$$

Note that the fulfillment of expressions (8) is easy to check using the methods described in [12. pp. 7–10].

Since the requirements for the quality of control are expressed in root quality indicators, the controller will be calculated according to the modal control scheme. Following the method of synthesizing a modal controller (described, for example, in the monograph [6. pp. 5–20]), the controller for the nominal model (7) is sought in the form of a difference equation of the n–1 order:

$$\beta(n-1,z)u(t)=\alpha(n-1,z)y(t)+\chi(n-1,z)g(t), \quad \beta_{n-1}=1, \tag{9}$$

here g is a given program signal. In the modal control scheme, the coefficients of the operators β(n – 1, z) and α(n – 1, z) of the controller (9) are calculated from the condition of conversion to the identity of the equation

$$a^{\text{st}}(2n-1,s)=a^{0}(n,\mathbf{a}^{0},s)\beta(n-1,s)-b^{0}(m,\mathbf{b}^{0},s)\alpha(n-1,s),$$

$$a^{\text{st}}_{2n-1}=1, \tag{10}$$

where $a^{\text{st}}(2n-1, s)$ is the given characteristic polynomial of the standard control system (hereinafter referred to as the "standard"); the choice of standard is limited by the condition

$$\Lambda(a^{\text{st}})\subset \operatorname{int} S. \tag{11}$$

It is obvious that the choice of the polynomial χ(n – 1, s) does not affect the properties of robust stability and the robust quality of control, so the issue of calculating the polynomial χ(n – 1, s) is not considered in this paper.

The method for calculating the coefficients of polynomials β(n – 1, s) and α(n – 1, s) is given in [3. pp. 8–21], it is reduced to solving a system of 2n – 1 linear algebraic equations with respect to 2n – 1 unknown coefficients of polynomials β(n – 1, s) and α(n – 1, s). This system is uniquely solvable if the zeros of the polynomial $a^0(n, s)$ do not coincide with the zeros of the polynomial $b^0(m, s)$.

After closing the original object (6) by the controller synthesized according to the scheme (9)–(11), it is easy to obtain the following expression for the characteristic polynomial of the closed-loop system

$$a^{\text{c}}(2n+l-1,A,B,V,W,s)=\{a^{\text{c}}(2n+l-1,\mathbf{a},\mathbf{b},\mathbf{v},\mathbf{w},s):$$

$$\forall\mathbf{a}\in A,\mathbf{b}\in B,\mathbf{v}\in V,\mathbf{w}\in W\}, \tag{12}$$

here

$$a^{\text{c}}(2n+l-1,\mathbf{a},\mathbf{b},\mathbf{v},\mathbf{w},s)\doteq v(l,\mathbf{v},s)\cdot a(n,\mathbf{a},s)\cdot\beta(n-1,s)-$$

$$-w(h,\mathbf{w},s)\cdot b(m,\mathbf{b},s)\cdot\alpha(n-1,s).$$

The set of polynomials $a^{\text{c}}(2n + l - 1, A, B, V, W, s)$ is called the "family of characteristic polynomials" of a closed-loop system. A set of

$$\Lambda(a^{\text{c}})=\{\lambda_i: \quad \exists\mathbf{a}\in A,\exists\mathbf{b}\in B,\exists\mathbf{v}\in V,$$

$$\exists\mathbf{w}\in W, \quad a^{\text{c}}(2n+l-1,\mathbf{a},\mathbf{b},\mathbf{v},\mathbf{w},\lambda_i)=0, \quad i=\overline{1,2n+l-1}\}$$

we will call the "set of zeros" of the family of characteristic polynomials (or the "set of poles" of the PF) of a closed-loop system.

We will assume that a closed system with characteristic polynomial (12) has a robust quality of control if the set $\Lambda(a^{\text{c}})$ lies inside the area S, that is, the condition is satisfied (hereinafter, **the condition of robust quality of control**)

$$\Lambda(a^{\text{c}})\subset \operatorname{int} S. \tag{13}$$

In the presence of structural disturbances in the control object, it is impossible to guarantee in advance that the modal controller calculated by formulas (9) – (11) will ensure the fulfillment of condition (13). Thus, the problem of synthesizing a modal controller in the presence of structural disturbances in the control object consists of the following steps:

1) synthesis of a modal controller for a nominal object (7) (according to formulas (9)–(11));

2) subsequent verification of the fulfillment of condition (13) for a given family of characteristic polynomials (12) (**the problem of testing the robust quality of control**).

The next section presents a numerical method for checking the robust quality of control for a family of characteristic polynomials of the form (12).

2. Numerical method for checking the robust quality of control

2.1. Robust control quality criterion

Here and below (where this will not cause misunderstandings), we will omit the arguments of polynomials that are not essential for reasoning.

Theorem 1. *Let the family of polynomials* (12) *satisfy conditions* (8), (11). *Then, in order for this family to satisfy the condition of robust control quality* (13), *it is necessary and sufficient that*

$$0 \notin A^{c}(s), \quad \forall s \in \partial S, \tag{14}$$

where set

$$A^{c}(s) \doteq \left\{ a^{c}(2n+l-1, s) : \forall \mathbf{a} \in A, \mathbf{b} \in B, \mathbf{v} \in V, \mathbf{w} \in W \right\} \subset C^{1},$$

is the "geometric image" of the family of polynomials (12) *for the point* $s \in \partial S$.

Proof. Polynomial

$$a^{c.0}\left(2n+l-1, \mathbf{a}^{0}, \mathbf{b}^{0}, \mathbf{v}^{0}, \mathbf{w}^{0}, s\right) \doteq v\left(l, \mathbf{v}^{0}, s\right) \cdot a\left(n, \mathbf{a}^{0}, s\right) \times$$
$$\times \beta(n-1, s) - w\left(h, \mathbf{w}^{0}, s\right) \cdot b\left(m, \mathbf{b}^{0}, s\right) \cdot \alpha(n-1, s) \tag{15}$$

belongs to the family (12). By conditions (8) and (11), all zeros of polynomial (15) lie inside the given area *S*. All polynomials of family (12) have the same degree and the coefficient at the highest degree is equal to 1, therefore, an arbitrary polynomial of family (12) can be considered as the result variations of the coefficients ($\mathbf{a} \in A$, $\mathbf{b} \in B$, $\mathbf{v} \in V$ and $\mathbf{w} \in W$) with respect to the coefficients of the polynomial (15).

A change in the number of zeros of the polynomial lying inside the area *S* can occur if, as a result of the variation of the coefficients, at least one zero of the polynomial crosses the boundary of the area *S* and condition (14) is violated.

Thus, the fulfillment of conditions (8), (11), and (14) is necessary and sufficient for the robust control quality condition (13) to be satisfied. *Q.E.D.*

The polynomials in the family (12) contain products of variable parameters; therefore, the set $A^{c}(s)$ on C^{1} can be non-convex (and even non-simply connected). Thus, the direct construction of the nonconvex set $A^{c}(s)$ and the subsequent verification of expression (14) are difficult problems. Below we obtain sufficient conditions for robust control performance by estimating the distance from the set $A^{c}(s)$ to the point (0, 0*j*).

For a point $s \in \partial S$, this distance is (taking into account notation (1)):

$$\left| a^{c}\left(2n+l-1, \mathbf{a}, \mathbf{b}, \mathbf{v}, \mathbf{w}, s\right) \right|_{-},$$

thus, expression (14) takes the form

$$\left| a^{c}\left(2n+l-1, \mathbf{a}, \mathbf{b}, \mathbf{v}, \mathbf{w}, s\right) \right|_{-} > 0, \quad \forall s \in \partial S.$$

The expression on the left side of the last inequality satisfies the estimate

$$\left| a^{c}\left(2n+l-1, \mathbf{a}, \mathbf{b}, \mathbf{v}, \mathbf{w}, s\right) \right|_{-} = \Big| \left(v^{0}\left(l, \mathbf{v}^{0}, s\right) + \delta v(l, \delta\mathbf{v}, s) \right) \times$$
$$\times \left(a^{st}(2n-1, s) + \delta a(n-1, \delta\mathbf{a}, s) \cdot \beta(n-1, s) - \delta b(m, \delta\mathbf{b}, s) \times \right.$$
$$\left. \times \alpha(n-1, s) \right) + \left(v^{0}\left(l, \mathbf{v}^{0}, s\right) + \delta v(l, \delta\mathbf{v}, s) - w^{0}\left(h, \mathbf{w}^{0}, s\right) - \delta w(h, \delta\mathbf{w}, s) \right) \times$$
$$\times \left(b^{0}\left(m, \mathbf{b}^{0}, s\right) + \delta b(m, \delta\mathbf{b}, s) \right) \cdot \alpha(n-1, s) \Big|_{-} \geq \Big| v^{0}\left(l, \mathbf{v}^{0}, s\right) +$$
$$+ \delta v(l, \delta\mathbf{v}, s) \Big|_{=} \cdot \Big| a^{st}(2n-1, s) + \delta a(n-1, \delta\mathbf{a}, s) \cdot \beta(n-1, s) -$$
$$- \delta b(m, \delta\mathbf{b}, s) \cdot \alpha(n-1, s) \Big|_{-} - \Big| v^{0}\left(l, \mathbf{v}^{0}, s\right) + \delta v(l, \delta\mathbf{v}, s) -$$
$$- w^{0}\left(h, \mathbf{w}^{0}, s\right) - \delta w(h, \delta\mathbf{w}, s) \Big|_{+} \cdot \Big| b^{0}\left(m, \mathbf{b}^{0}, s\right) + \delta b(m, \delta\mathbf{b}, s) \Big|_{+} \cdot \left| \alpha(n-1, s) \right|.$$

The above reasoning can be considered as a non-rigorous proof of the following theorem.

Theorem 2. *Let the family of polynomials* (12) *satisfy conditions* (8), (11) *and*

$$\rho(s) \doteq \rho_1(s) - \rho_2(s) > 0, \quad \forall s \in \partial S, \tag{16}$$

where

$$\rho_1(s) \doteq \left| v^0\left(l, \mathbf{v}^0, s\right) + \delta v\left(l, \delta\mathbf{v}, s\right) \right|_- \cdot \left| a^{\text{st}}\left(2n-1, s\right) + \right.$$
$$\left. + \delta a\left(n-1, \delta\mathbf{a}, s\right) \cdot \beta\left(n-1, s\right) - \delta b\left(m, \delta\mathbf{b}, s\right) \cdot \alpha\left(n-1, s\right) \right|_-,$$
$$\rho_2(s) \doteq \left| v^0\left(l, \mathbf{v}^0, s\right) + \delta v\left(l, \delta\mathbf{v}, s\right) - w^0\left(h, \mathbf{w}^0, s\right) - \delta w\left(h, \delta\mathbf{w}, s\right) \right|_+ \times$$
$$\times \left| b^0\left(m, \mathbf{b}^0, s\right) + \delta b\left(m, \delta\mathbf{b}, s\right) \right|_+ \cdot \left| \alpha\left(n-1, s\right) \right|. \tag{17}$$

Then the family of polynomials (12) *satisfies* (13).

The technology for checking condition (16) is described below.

2.2. Method for checking the robust quality of control

The calculation of the functions $\rho_1(s)$ and $\rho_2(s)$ at the point s is reduced to solving 4 quadratic programming problems (hereinafter referred to as QP problems)

$$J_k\left(\mathbf{x}_k\right) = \frac{1}{2}\mathbf{x}_k^{\text{T}}\mathbf{H}_k\mathbf{x}_k + \mathbf{q}_k^{\text{T}}\mathbf{x}_k + e_k \to \min, \quad k \in \overline{1,4},$$
$$\mathbf{\Xi}_k\mathbf{x}_k - \zeta_k \le \mathbf{0}, \tag{18}$$

where

$$\mathbf{x}_1 \doteq \delta\mathbf{v}, \quad \Delta\mathbf{x}_1 \doteq col\left(\Delta v_1, \cdots, \Delta v_l\right), \quad \xi_1 \doteq col\left(s, \cdots, s^l\right),$$
$$\boldsymbol{\varphi}_1 \doteq \text{Re}\left(\xi_1\right), \quad \boldsymbol{\psi}_1 \doteq \text{Im}\left(\xi_1\right),$$
$$\theta_1 \doteq v^0\left(l, \mathbf{v}^0, s\right), \quad \mathbf{x}_2 \doteq col\left(\delta\mathbf{a}, \quad \delta\mathbf{b}\right),$$
$$\Delta\mathbf{x}_2 \doteq col\left(\Delta a_0, \cdots, \Delta a_{n-1}, \Delta b_0, \cdots, \Delta b_m\right),$$
$$\xi_2 \doteq col\left(\beta\left(n-1, s\right), \cdots, s^{n-1} \cdot \beta\left(n-1, s\right), \right.$$
$$\left. -\alpha\left(n-1, s\right), \quad \cdots, \quad -s^m \cdot \alpha\left(n-1, s\right)\right),$$
$$\boldsymbol{\varphi}_2 \doteq \text{Re}\left(\xi_2\right), \quad \boldsymbol{\psi}_2 \doteq \text{Im}\left(\xi_2\right), \quad \theta_2 \doteq a^{\text{st}}\left(2n-1, s\right),$$
$$\mathbf{x}_3 \doteq col\left(\delta\mathbf{v}, \quad \delta\mathbf{w}\right), \quad \Delta\mathbf{x}_3 \doteq col\left(\Delta v_1, \cdots, \Delta v_l, \Delta w_1, \cdots, \Delta w_h\right),$$
$$\xi_3 \doteq col\left(1, \quad s, \quad \cdots, \quad s^l, \quad -1, \quad -s, \quad \cdots, \quad -s^h\right),$$
$$\boldsymbol{\varphi}_3 \doteq \text{Re}\left(\xi_3\right), \quad \boldsymbol{\psi}_3 \doteq \text{Im}\left(\xi_3\right),$$
$$\theta_3 \doteq v^0\left(l, \mathbf{v}^0, s\right) - w^0\left(h, \mathbf{w}^0, s\right), \quad \mathbf{x}_4 = \delta\mathbf{b},$$
$$\Delta\mathbf{x}_4 \doteq col\left(\Delta b_0, \quad \cdots, \quad \Delta b_m\right), \quad \xi_4 \doteq col\left(1, \quad s, \quad \cdots, \quad s^m\right),$$
$$\boldsymbol{\varphi}_4 \doteq \text{Re}\left(\xi_4\right), \quad \boldsymbol{\psi}_4 \doteq \text{Im}\left(\xi_4\right), \quad \theta_4 \doteq b^0\left(m, \mathbf{b}^0, s\right),$$
$$\mathbf{H}_k = \begin{cases} +2\left(\boldsymbol{\varphi}_k \cdot \boldsymbol{\varphi}_k^{\text{T}} + \boldsymbol{\psi}_k \cdot \boldsymbol{\psi}_k^{\text{T}}\right), & k \in \overline{1,2}, \\ -2\left(\boldsymbol{\varphi}_k \cdot \boldsymbol{\varphi}_k^{\text{T}} + \boldsymbol{\psi}_k \cdot \boldsymbol{\psi}_k^{\text{T}}\right), & k \in \overline{3,4}, \end{cases}$$
$$\mathbf{q}_k = \begin{cases} +2\left(\text{Re}\left(\theta_k\right) \cdot \boldsymbol{\varphi}_k + \text{Im}\left(\theta_k\right) \cdot \boldsymbol{\psi}_k\right), & k \in \overline{1,2}, \\ -2\left(\text{Re}\left(\theta_k\right) \cdot \boldsymbol{\varphi}_k + \text{Im}\left(\theta_k\right) \cdot \boldsymbol{\psi}_k\right), & k \in \overline{3,4}, \end{cases}$$
$$e_k = \text{Re}^2\left(\theta_k\right) + \text{Im}^2\left(\theta_k\right),$$

and Ξ_k and ζ_k are block matrices and vectors

$$\Xi_k = \left(\mathbf{E} \quad | -\mathbf{E}\right)^{\mathrm{T}}, \quad \zeta_k = col\left(\Delta\mathbf{x}_k, \quad \Delta\mathbf{x}_k\right), \quad k \in \overline{1,4}.$$

For calculate Ξ_k the dimension of the matrix **E** must be coordinated with the dimension of the vector $\mathbf{x}_k$.

It is easy to verify that

$$\mathbf{H}_k^{\mathrm{T}} = \mathbf{H}_k, \quad e_k \geq 0, \quad k \in \overline{1,4}.$$

In notation (18), the functions $\rho_1(s)$ and $\rho_2(s)$ take the form:

$$\rho_1(s) = J_1^{\frac{1}{2}}\left(\mathbf{x}_1^*\right) \cdot J_2^{\frac{1}{2}}\left(\mathbf{x}_2^*\right), \qquad \rho_2(s) = J_3^{\frac{1}{2}}\left(\mathbf{x}_3^*\right) \cdot J_4^{\frac{1}{2}}\left(\mathbf{x}_4^*\right) \cdot \left|\alpha(n-1,s)\right|, \tag{19}$$

$$\mathbf{x}_k^* \doteq \arg\min_{\Xi_k\mathbf{x}_k - \zeta_k \leq \mathbf{0}} f_k\left(\mathbf{x}_k\right), \qquad f_k\left(\mathbf{x}_k\right) \doteq \frac{1}{2}\mathbf{x}_k^{\mathrm{T}}\mathbf{H}_k\mathbf{x}_k + \mathbf{q}_k^{\mathrm{T}}\mathbf{x}_k. \tag{20}$$

Problem QP (20) is among the solved ones. In the case of constraints in the form of linear inequalities, this problem can be solved only by numerical methods.

Further, for convenience of presentation, we assume that the boundary ∂S is replaced by a finite number of points

$$\Omega = \left\{s_i \in \partial S, \quad i \in \overline{1, N}\right\}, \qquad \left(N < \infty\right),$$

here the points s_i included in the set Ω are chosen on the boundary ∂S based on the condition:

$$\max_{s\in\partial S} \Delta\arg\left(\omega - s\right)\Big|_{\omega=s_i}^{\omega=s_{i+1}} \leq \frac{1}{2n+l-1} \cdot \frac{\pi}{2} \text{ radian.}$$

Based on the above considerations, we can formulate the following algorithm for checking the robust quality of control.

2.3. The algorithm for checking the robust quality of control

Step 1. At each point $s_i \in \Omega(N)$ solve four QP problems (18), and use formulas (19)–(20) to calculate the values of $\rho_1(s)$ and $\rho_2(s)$.

Step 2. Check the fulfillment of condition (16). If condition (16) is satisfied, then for a given family of characteristic polynomials, the robust quality of control is satisfied; if not, the modal controller should be calculated using the complete model of the control object (6).

3. Example of checking the robust quality of control

Let the control object be given by a model of the form:

$$\left(v^0(2,z) + \Delta v(2,z)\right) \cdot \left(a^0(2,z) + \Delta a(1,z)\right) y(t) =$$
$$= \left(w^0(2,z) + \Delta w(2,z)\right) \cdot \left(b^0(1,z) + \Delta b(1,z)\right) u(t),$$

here

$$a^0(2,z) = z^2 - 1{,}918z + 0{,}923, \qquad b^0(1,z) = 0{,}232z - 0{,}179,$$
$$\Delta a(1,z) = 10^{-4}[-1;1]z + 10^{-4}[-2;2],$$
$$\Delta b(1,z) = 10^{-2}[-1;1]z + 10^{-2}[-1;1],$$
$$v^0(2,z) = 1{,}830z^2 - 2{,}707z + 1,$$
$$w^0(2,z) = 1{,}848z^2 - 2{,}717z + 1,$$
$$\Delta v(2,z) = 10^{-4}[-5;5]z^2 + 10^{-3}[-1;1]z,$$
$$\Delta w(2,z) = 10^{-4}[-5;5]z^2 + 10^{-3}[-1{,}5;1{,}5]z.$$

Zeros of the polynomials $v^0(2, z)$ and $w^0(2, z)$:

$$\Lambda\left(v^0\right) = \{0{,}715, \quad 0{,}764\} \subset C^1,$$

$$\Lambda\left(w^0\right)=\{0,735+0,027\,j,\quad 0,735-0,027\,j\}\subset C^1.$$

Requirements for the quality of management are set by the area

$$S=\left\{s:\eta_2\le|s|\le\eta_1,\quad |\arg(s)|\le\varphi\right\},$$

where $\eta_1 = 0,607$, $\eta_2 = 0,961$, $\varphi = \pi/4$ radian. Note that the sets $\Lambda(v^0)$ and $\Lambda(w^0)$ lie inside the area S.

We choose the characteristic polynomial of the standard

$$a^{\text{st}}(3,z)=z^3-2,715z^2+2,456\,z-0,741.$$

For the nominal model of control object and the standard $a^{\text{st}}(3, z)$, the modal controller is calculated

$$(z-0,833)u(t)=(-0,158z+0,157)\,y(t)+\chi_0 g(t).$$

The family of characteristic polynomials of a closed-loop system is

$$\begin{aligned}a^{\text{c}}(5,s)=&\left(v^0(2,s)+\Delta v(2,s)\right)\times\\ &\times\left(a^{\text{st}}(3,s)+\Delta a(1,s)\cdot\beta(1,s)-\Delta b(1,s)\cdot\alpha(1,s)\right)+\\ &+\left(v^0(2,s)+\Delta v(2,s)-w^0(2,s)-\Delta w(2,s)\right)\times\\ &\times\left(b^0(1,s)+\Delta b(1,s)\right)\cdot\alpha(1,s).\end{aligned}\tag{21}$$

To check the properties of the robust quality of control on the boundary of the area S, 42 points were chosen, starting from the point $(0,961 + 0j)$ at an equal distance from one another. The smallest value of the function $\rho(s)$ is $6,136\cdot10^{-4}$, it is reached at the point $(0,607 + 0j)$. Since the value $\rho = 6,136\cdot10^{-4}$ is greater than zero, therefore, according to Theorem 2, the family of polynomials (21) has a robust control quality.

Conclusions

In this article, a criterion for the robust quality of control is obtained for a closed-loop system consisting of a linear one-dimensional control object with structural disturbances and a modal controller. Theorem 2 establishes a criterion for the set of zeros to belong to the family of characteristic polynomials of a given area S on the complex plane.

Based on this theorem, a numerical method for checking the robust quality of control has been developed, which consists in solving four quadratic programming problems for each point s on the boundary of the area S.

The function ρ, defined by formulas (16)–(17), makes it possible to compare various closed systems with each other. Thus, the function ρ is a measure of robust properties, which can be further used to formulate the problem of improving the robust quality of control, for example, using the freedom to choose the characteristic polynomial of the standard $a^{\text{st}}(2n–1, s)$ within the limits of condition (11).

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