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Solution of the parameter problem of the Schwarz–Christoffel conformal mapping of the interior (exterior) of a circle onto the interior (exterior) of a polygon

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Abstract. We propose an analytical solution of the parameter problem of the Schwarz–Christoffel conformal mapping of the interior (exterior) of a circle onto the interior (exterior) of a polygon by use of the behavior of the Newtonian simple layer and logarithmic potentials equal to a constant inside of a simply connected domain.
Keywords: conformal mapping, Schwarz–Christoffel parameter problem, potential method

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Научная статья

Решение проблемы параметров Кристоффеля–Шварца конформного отображения внутренности (внешности) круга на внутренность (внешность) многоугольника

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Аннотация. Приведено аналитическое решение проблемы параметров Кристоффеля–Шварца конформного отображения внутренности (внешности) круга на внутренность (внешность) многоугольника с использованием свойств ньютоновского простого слоя и логарифмического потенциалов, равных константе внутри односвязной области.
Ключевые слова: конформное отображение, проблема параметров Кристоффеля–Шварца, метод потенциала

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Introduction

In this paper, the potential method [1, 2] which was popular in Russia in 1920s–1960s, is used for finding the solution of the problem indicated in the heading. As far as the author knows, the problem has no analytical solution [3–10]. The main results of [1] in a brief presentation with modern notation are given in the first section [11]. Some comments about the method will be made in conclusion.

1. Newtonian potential equal to a constant inside of a simply connected three-dimensional domain

A Newtonian source $1/r(p,q)$, where $r(p,q)$ is the distance between the points $q(y_1,y_2,y_3)$ and $p(x_1,x_2,x_3)$, $r=\sqrt{\sum_1^3 (x_i-y_i)^2}$, is a harmonic function of p in the space $\mathbb{R}^3$, with the exception of the point of source q, that is, in $p\in\mathbb{R}^3\setminus\{q\}$. The function

$$V(p,\varphi)=\frac{1}{2\pi}\int_S \frac{\varphi(q)}{r(p,q)}dS_q, \tag{1}$$

where φ is the density function and S is a two-dimensional surface in $\mathbb{R}^3$, is commonly referred as a Newtonian potential of the simple layer [1, 2]. If S is the boundary of a simply connected domain Θ, (1) is a harmonic function in Θ or in $\mathbb{R}^3\setminus\Theta$. The potential (1) has the limit values of the normal derivative of external normal n_p at the point $p\in S$, $S\in C_1$, from the inside (index plus) and from the outside (index minus) of the domain Θ:

$$\left[\frac{\partial V(p,\varphi)}{\partial n_p}\right]^{\pm}=\pm\varphi(p)+\frac{1}{2\pi}\int_S \frac{\partial}{\partial n_p}\left(\frac{\varphi(q)}{r(p,q)}\right)dS_q, \tag{2}$$

where the integral is singular and exists in the sense of the principal value.

The potential (1) satisfies the condition of radiation; therefore, the representation of a harmonic function in $\mathbb{R}^3\setminus\Theta$ obtained by use of it and satisfying the condition of radiation is unique. As the integral in (1) has a weak singularity, (1) is a continuous function in $\mathbb{R}^3$; therefore, the representation by the potential (1) of a harmonic function in Θ is also unique because any unique representation (1) of a solution of the Dirichlet problem in $\mathbb{R}^3\setminus\Theta$ corresponds with unique representation (1) of a solution of the Dirichlet problem in Θ with the same boundary values on S.

Therefore, there is the unique function φ_0, $\int_S \varphi_0^2 dS=1$, at which the potential (1) $V(p,\varphi_0)$ is equal to a constant inside of the simply connected domain with the boundary $S\in C_1$. In the general case, the form of φ_0 is not known; in the particular case, when S

is the surface of a full sphere, φ_0 is a constant c_0, easily calculated through the value of the potential in the center of the full sphere:

$$2\hat{R}c_0\ , \tag{3}$$

where $\hat{R}$ is the radius.

The expressions (4) follow from (2).

$$\left[\frac{\partial V(p,\varphi_0)}{\partial n_p}\right]^+ = 0, \quad \left[\frac{\partial V(p,\varphi_0)}{\partial n_p}\right]^- = -2\varphi_0. \tag{4}$$

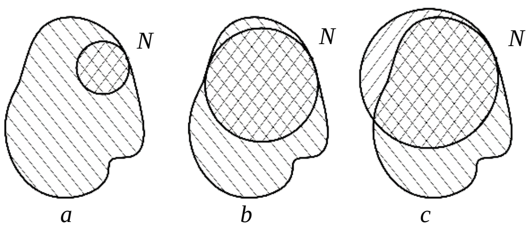

Fig. 1. Definition of the harmonic function of the difference of two potentials

If we consider the harmonic function defined by the difference of two potentials (1) in which one of them is specified on the smooth boundary of a simply connected domain and the other is specified on the surface of the full sphere located inside of this domain, when both potentials are equal to the same constant inside this domain and inside this full sphere, we obtain the limit expressions of the normal derivative from outside of the domain at the point *N* (Fig. 1a):

$$-2\varphi_0(N) + 2c_0 \geq 0, \tag{5}$$

since the harmonic function outside the domain has the smallest values equal to zero at the tangency point *N* and at the infinitely distant boundary.

Indeed, the potential (1) with density $\varphi = c_0$, $c_0 > 0$, specified at the full sphere's boundary, is equal inside the full sphere to the value in the center (3), and monotonically decreases to zero from the boundary of this full sphere towards any infinitely distant point. Therefore, at all points of the boundary of the domain (Fig. 1a), with the exception of the point *N*, the values of the potential specified on the boundary of the full sphere are less than (3) but larger than zero, and at the point *N* the value is equal to (3). Therefore, taking into account (4), condition (5) is satisfied since the difference of the potentials on the boundary of the domain is greater than zero or equal to it and at the infinitely distant boundary is zero, and this difference, being a harmonic function outside the domain, reaches the maximum and minimum values at the boundary of the definitional domain and cannot have negative values. Therefore, the limit value of the normal derivative of the potential's difference at the point *N* from outside of the domain (Fig. 1a) cannot be negative (5).

Let us increase the radius of the full sphere so that the tangency at the point *N* is preserved and the potential specified at its boundary is equal to the same constant inside the full sphere. Then, the value of c_0 will decrease inversely to the ratio of current and original radiuses (3), the second item of the sum (5) will decrease. Starting with a certain value of the radius, inequality (5) will cease to be satisfied. This is possible if the full sphere ceases to be completely located in the domain (Fig. 1c), the potential's difference will have negative values on the boundary of the union of the domain and the full sphere at the points of the full sphere outside of the domain. Therefore, for the inscribed full sphere the inequality (5) goes over into the equality (Fig. 1b). By this equality we can determine φ_0 through sequentially inscribing of a full sphere at a point of the boundary of the domain, by the value of the radius of the inscribed full sphere and the value of the given constant equal to the potentials.

The given algorithm for determining the values of φ_0 for a simply connected domain with a smooth boundary, $S \in C_1$, can be used for a piecewise-smooth boundary as it is applicable in the process of approach of rounding radius at unregular (angular) points of a piecewise-smooth boundary to zero. The potential obtained in this way will be equal to a given constant at all points of the simply connected domain with a piecewise-smooth boundary but will not have representation (1) at some irregular points of the boundary, at which the integral in (1) diverges due to the presence of singularity of the φ_0. (Since the values of φ_0 are inversely proportional to the radius of the inscribed full sphere, $|\varphi_0| \to \infty$, if the radius tends to zero. At some such points, the degree of singularity of φ_0 allows us to calculate (1).)

(The function φ_0 is used in physics. If we will locate the electrical charges interacting according to Coulomb's law on an electric conductor, they will be distributed on the surface of the electric conductor in accordance with the density φ_0 with an accuracy to a coefficient, generating a constant electrical potential inside the conductor. The analysis carried out in this section shows that the electric charges on the electrical conductor occupying a simply connected domain with a piecewise smooth boundary S in vacuum cannot be in a stable state if there is no integral $\int_S \varphi_0(p)dS_p$ (if it diverges). They will flow out from the conductor until the total amount of the charges will be zero.)

2. Logarithmic potential equal to a constant inside of a simply connected two-dimensional domain

The algorithm given in the previous section for the Newtonian potential in the three-dimensional case cannot be repeated for the logarithmic potential (6) because it does not satisfy the radiation condition in the two-dimensional case and does not approach zero at the infinitely distant boundary:

$$\bar{V}(p,\varphi) = \frac{1}{\pi}\int_S \ln\left(\frac{1}{r(p,q)}\right)\varphi(q)dS_q, \tag{6}$$

where $r(p,q)$ is the distance between the points $q(y_1, y_2)$ and $p(x_1, x_2)$, $r = \sqrt{\sum_1^2 (x_i - y_i)^2}$.

Let us use the three-dimensional problem for generalization for the two-dimensional problem. Since the potential $V(p,\varphi_0)$ is equal to a constant inside of the simply connected domain, the section of a body elongated along the axis x_3 with a constant cross section corresponds to the two-dimensional problem in the plane Ox_1x_2 (Fig. 2).

Let us consider the integration element of the infinitely elongated domain with a constant cross section (Fig. 2):

$$dS_{3q} = r(p,q)d\alpha / \cos(\alpha)dL. \tag{7}$$

Let the point $p(z_1, z_2, 0)$ be the observation point, $q(y_1, y_2, y_3)$ be the integration point, $a(a_1, a_2, a_3)$ be the difference of coordinates: $a_i = z_i - y_i$, $i = 1,2,3$, and $q_p(y_1, y_2, 0)$ be the intermediate point (Fig. 3). In Fig. 3, dL is the element of integration along the perimeter of the cross section in Ox_1x_2. The angle α is located in a plane parallel to the axis x_3 passing through the points p, q and is measured from the plane Ox_1x_2.

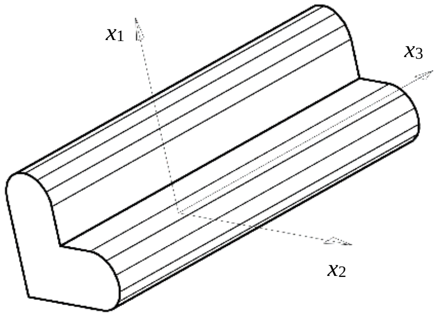

Fig. 2. Domain elongated along the axis x_3 with a constant cross section

Let us consider expression (2) equal to a zero constant in the domain (Fig. 2). Since the values of φ_0 found by the algorithm described in the previous section are the same in each transverse section, parallel to the plane Ox_1x_2, far from the L-shaped ends, and the influence of the integrals of L-shaped ends at the points of the plane Ox_1x_2 approaches zero at the approach of the extended along Ox_3 size infinity, we can consider expression (2) for the potential equal to a constant inside the domain (Fig. 2), when its size extended along the axis x_3 approaches infinity. In this expression, the integral over the surface extended along the axis x_3 will remain only, let us denote this surface as S_3, the integrals of the L-shaped ends will become zero.

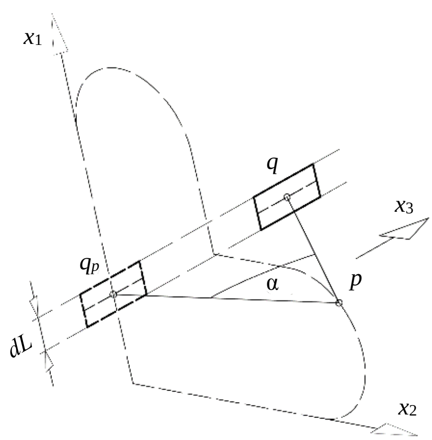

Fig. 3. Element of integration $dS_{3q} = r(p,q)d\alpha / \cos(\alpha)dL$

The expression corresponds to the element of integration over the infinite surface S_3 of the domain infinitely extended along the axis x_3 (Fig. 2):

$$\frac{\partial}{\partial n_p}\left(\frac{\varphi_0(q)}{r(p,q)}\right)dS_{3q} = -\frac{a_1 n_1 + a_2 n_2}{r(p,q)^2\cos(\alpha)}\varphi_0(q)d\alpha dL =$$

$$= -\frac{(a_1 n_1 + a_2 n_2)\cos(\alpha)}{r(p,q_p)^2}\varphi_0(q_p)d\alpha dL,$$

where (7) and the equality $r(p,q) = r(p,q_p)/\cos(\alpha)$ are taken into account (Fig. 3). (When the points *p* and *q* are located on the straight line parallel to Ox_3, their two other coordinates coincide, and the expression in the brackets is equal to zero, hence the contribution to the integral sum is zero.) Therefore, the integral in (2) on the infinite surface S_3 infinitely extended along the axis x_3 (Fig. 2) is equal to

$$\int_{S_3}\frac{\partial}{\partial n_p}\left(\frac{\varphi_0(q)}{r(p,q)}\right)dS_{3q} = -\int_{-\pi/2}^{\pi/2}\int_L \frac{(a_1 n_1 + a_2 n_2)\cos(\alpha)}{r(p,q_p)^2}\varphi_0(q_p)dLd\alpha =$$

$$= -\sin(\alpha)\big|_{-\pi/2}^{\pi/2}\int_L \frac{a_1 n_1 + a_2 n_2}{r(p,q_p)^2}\varphi_0(q_p)dL = -2\int_L \frac{a_1 n_1 + a_2 n_2}{r(p,q_p)^2}\varphi_0(q_p)dL. \quad (8)$$

It is easy to see that the last expression (8) coincides with the expression of the integral of the normal derivative of the logarithmic potential (6). Thus, in the limit, we have received the expressions of the limiting values of the normal derivative of the logarithmic potential (6) at the boundary of the two-dimensional domain, which coincides with the cross section of the three-dimensional domain (Fig. 2):

$$\lim_{S_3\to\infty}\left[\frac{\partial V(p,\varphi_0)}{\partial n_p}\right]^{\pm} = \left[\frac{\partial \overline{V}(p,\varphi_0)}{\partial n_p}\right]^{\pm} = \pm\varphi_0(p) + \frac{1}{\pi}\int_S \frac{\partial}{\partial n_p}\ln\left(\frac{1}{r(p,q)}\right)\varphi_0(q)dS_q =$$

$$= \pm\varphi_0(p) - \frac{1}{\pi}\int_L \frac{a_1 n_1 + a_2 n_2}{r(p,q_p)^2}\varphi_0(q_p)dL, \quad (9)$$

where the integral exists in the sense of the principal value at the points on the smooth parts of a piecewise-smooth boundary (the calculation of the summand outside the integral in (9) is similar to the calculation of $\frac{\varphi(p)}{\pi}\int_{S_R}\frac{d}{x_1^2 + d^2}dx_1$ in [12, p. 70]), that is,

$$\left[\frac{\partial \overline{V}(p,\varphi_0)}{\partial n_p}\right]^{+} = 0,\ \left[\frac{\partial \overline{V}(p,\varphi_0)}{\partial n_p}\right]^{-} = -2\varphi_0. \quad (10)$$

Since (10) were obtained by generalization of the three-dimensional case, the potential equal to a constant inside a simply connected two-dimensional domain has no representation (6) at some irregular points of the two-dimensional boundary where $|\varphi_0| \to \infty$.

Thus, in the two-dimensional case, we can use the algorithm for finding the density function φ_0 of the logarithmic potential $\overline{V}(p,\varphi_0)$ (6), equal to a constant inside of the two-dimensional simply connected domain with a piecewise-smooth boundary, similar to finding φ_0 in the three-dimensional case; instead of inscribed full spheres, we have to use inscribed circles.

3. Conformal mapping of the interior (exterior) of a circle onto the interior (exterior) of a polygon

Conformal mapping of the interior (exterior) of a circle onto the interior (exterior) of a polygon can be performed by the function [5, p. 179]

$$\varpi(z) = C\int_{z_0}^{z}(z-e_1)^{(\tilde{\gamma}_1-1)}(z-e_2)^{(\tilde{\gamma}_2-1)}\ldots(z-e_n)^{(\tilde{\gamma}_n-1)}dz + C_1, \tag{11}$$

where $\tilde{\gamma}_k$ are the interior (exterior) angles of the corners of the polygon measured in radians divided by π, $0 < \tilde{\gamma}_k < 2$; e_k are the points of the unit circle corresponding to the vertices of the polygon, $|e_k| = 1$; and z_0, C, C_1 are some constants.

In (9), from the equality of the derivative of the Newtonian potential to zero in the limiting case the equality to zero of the derivative of the logarithmic potential follows; however, we do not know the value of constant to which the logarithmic potential with the density φ_0 is equal inside of the two-dimensional simply connected domain of the cross section (Fig. 2). If we set the logarithmic potential with the unit density at the boundary of a unit circle, equal to its value in the center, inscribed in the domain of the form of a polygon with rounded corners (Fig. 4), and require that the logarithmic potential specified on the boundary of this domain has to be equal to the same constant, the densities of these potentials at the tangent point will not coincide. That is, the density of the potential φ_0 on the boundary of the domain at the tangent point will differ from the unit density at the boundary of the circle by the coefficient κ. At a different point of the boundary of the domain (Fig. 4), φ_0 will be equal to κ/*R*, where *R* is the radius of the inscribed circle at this point.

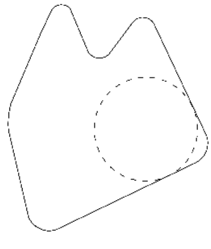

Fig. 4. Polygon with rounded corners

If we obtain the conformal mapping of the interior of the unit circle (Fig. 4) onto the interior of the domain (Fig. 4), the sources of the logarithmic potential at the boundary of the circle will be distributed along the boundary of the domain and generate inside the domain the same constant potential. The total amount of the sources before and after the mapping will be equal to 2π. That is,

$$\int_0^J \varphi_0(q)dL_q = \int_0^J \frac{\kappa}{R(q)}dL_q = \int_0^{2\pi} d\alpha = 2\pi,$$

where *J* is the perimeter of the boundary of the domain. This corresponds to the replacement of the integration variable:

$$dL_q = \frac{R(q)}{\kappa}d\alpha.$$

Consequently,

$$\int_0^L \frac{\kappa}{R(q)} dL_q = \alpha, \tag{12}$$

where L is the distance along the boundary of the considered simply connected domain, α is the angle of the sector of the unit circle corresponding to L before the mapping, and

$$\kappa = 2\pi / \int_0^J \frac{1}{R(q)} dL_q .$$

Let the boundary of the domain (Fig. 4) be divided into N elements with the beginning and the end in the middles of the adjacent polygon legs (N is equal to the number of vertices of the polygon). Thus, the element with the number i corresponding to the contour AA_1BC_1C of the polygon legs adjacent at the angle β_i with the length L_i (Fig. 5) corresponds to the integral

$$D_i = \int_0^{L_i} \frac{1}{R(q)} dL_q = \bar{c}_i + \delta_i \frac{1}{\sin(\beta_i/2)\cos(\beta_i/2)} \left(\ln\left(\frac{\hat{b}_{i1}}{a}\right) + \ln\left(\frac{\hat{b}_{i2}}{a}\right) \right) + \delta_i v_i + (1-\delta_i) g_i ,$$

where $\begin{cases} \delta_i = 1 \quad , \quad \beta_i < \pi, \\ \delta_i = 0 \quad , \quad \beta_i > \pi, \end{cases}$

the first summand $\bar{c}_i$ is the value of the integral over the arc A_1BC_1 (if $\beta_i < \pi$, $\bar{c}_i = \beta_i$); the second summand is the value of the integral over the parts of the segments of straight lines $[AA_1]$, $[C_1C]$, having symmetric by β_i part of the segment of the straight line on the adjacent polygon leg: $\hat{b}_{i1} \le b_{i1}$, $\hat{b}_{i2} \le b_{i2}$ (Fig. 5); the third summand v_i is the value of the integral over the part of $[AA_1]$ or $[C_1C]$ for which there is no the symmetric by β_i part of the straight line on the adjacent polygon leg[1]; the fourth summand is the value of the integral over $[AA_1]$ and $[C_1C]$ in the case $\beta > \pi$; the value of a is assumed to be the same for all elements.

Indeed, if the segment $[AA_1]$ has a symmetric segment on the adjacent polygon leg $\hat{b}_{i1} = b_{i1}$ (Fig. 5), it corresponds to the integral [13, p. 253–254] in the local coordinate system (Fig. 6):

$$\int_a^{\hat{b}_{i1}} f\left(x, y(x)\right)\left(1 + y'(x)^2\right)^{\frac{1}{2}} dx = \frac{1}{\sin(\beta_i/2)\cos(\beta_i/2)} \ln\left(\frac{\hat{b}_{i1}}{a}\right), \tag{13}$$

where $y(x)$ is the equation of the line $[AA_1]$: $y(x) = \tan(\beta_i/2)x$;

$$f\left(x, y(x)\right) = \frac{1}{R(q)} = \frac{1}{\sin(\beta_i/2)x} .$$

The expression for $[C_1C]$ is similar to (13), where $\hat{b}_{i1}$ is replaced by $\hat{b}_{i2}$, $\hat{b}_{i2} \le b_{i2}$ (Fig. 5).

[1] For the case shown in Fig. 5, the third summand is equal to zero. The third summand is larger than zero, for example, if the half of the length of one of the two considered adjacent polygon legs is larger than the length of the other.

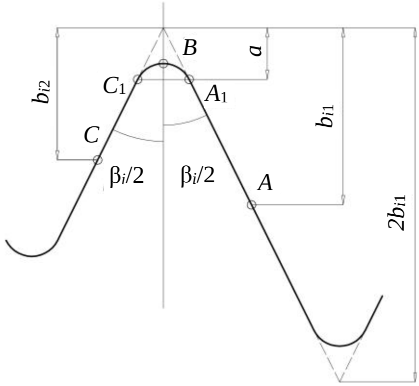

Fig. 5. The line AA_1BC_1C of the boundary of the polygon with rounded corners.

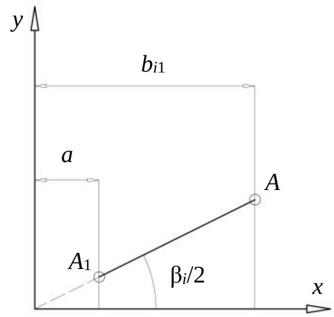

Fig. 6. The integration over the line AA_1 .

We get the coefficient $\kappa = 2\pi / \sum_{j=1}^{N} D_j$, and the value of the angle of the unit circle (12) corresponding to the line AA_1BC_1C is $\alpha_i = \kappa D_i$.

Let us consider the limit expression for α_i when $a \to 0$. Since at this approach $\ln(1/a) \to \infty$ and the values of $\bar{c}_i$, $\ln(\hat{b}_{i1})$, $\ln(\hat{b}_{i2})$, $\delta_i v_i$, $(1-\delta_i)g_i$ are finite, we obtain

$$\tilde{\alpha}_i = \lim_{a \to 0} \alpha_i = \lim_{a \to 0} \frac{D_i}{\sum_{j=1}^{N} D_j} 2\pi = 2\pi \frac{\delta_i}{\sin(\beta_i / 2)\cos(\beta_i / 2)} \Big/ \sum_{j=1}^{N} \delta_j \frac{1}{\sin(\beta_j / 2)\cos(\beta_j / 2)},$$

$$\sum_{i=1}^{N} \tilde{\alpha}_i = 2\pi, \;\; \tilde{\alpha}_i > 0 \;\text{ if }\; \beta_i < \pi, \;\; \tilde{\alpha}_i = 0 \;\text{ if }\; \beta_i > \pi.$$

As the integral (13) and the similar integral for $\hat{b}_{i2}$ at $\beta_i < \pi$ approach the same value tending towards infinity as $a \to 0$ when the rest summands in the expression of D_i are finite, the angular coordinates in radians on the unit circle correspond to the vertices of the polygon before the mapping:

$$\chi_1 = \tilde{\alpha}_1 / 2 \;,\;\; \chi_i = \chi_{i-1} + \tilde{\alpha}_{i-1} / 2 + \tilde{\alpha}_i / 2 \;,\;\; 2 \le i \le N\,.$$

These coordinates χ_i are the parameters of the Schwarz–Christoffel integral (11). Since we know these parameters, we can easily determine the values of the constants z_0, C, C_1. Thus, the parameter problem of the conformal mapping of the interior (exterior) of a circle onto the interior (exterior) of a polygon is solved.

Conclusion

The paper presents the solution by the method of potential of the parameter problem of the conformal mapping of the interior (exterior) of a circle onto the interior (exterior) of a polygon which has been sought for more than 150 years. This solution for the first time was published in [14] and discussed in [15].

The primary formulation of the potential method is given in the works of A.M. Lyapunov (1857–1918) [16]. It was created with the aim of finding the conditions for existence and uniqueness of solutions of the Dirichlet and Neumann problems, there was no purpose to solve the problems numerically at that time. Therefore, the restrictions that the considered boundary conditions have to be the Dirichlet conditions or the Neumann conditions on a smooth boundary of a simply connected domain seemed not very rigid. With the advent of available computers, these conditions have become too rigid for numerical calculations in which the boundary conditions are usually mixed, and the greatest interest is the calculations of the points in which the gradient of the required solution has a singularity – irregular points of a piecewise-smooth boundary and points at which the boundary conditions change their type. In the two-dimensional case, these points could be considered by mapping of the solution for a half-plane onto a wedge; in the three-dimensional case, there was no such algorithm. This led to a gradual extinction of interest in the potential method as a numerical tool.

The paper [12] contains the previously unknown form of a harmonic function in spherical coordinates and its representation by the potentials of a double or simple layer, which allows us to use the potential method for numerical calculations of the Dirichlet, Neumann, and mixed Dirichlet–Neumann problems on a piecewise-smooth boundary. That is, the main drawback of the traditional formulation of the method is eliminated. The author hopes to show that other reasons of unpopularity of the potential method can be eliminated also.

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