Asymptotic Analysis of Intensity of Poisson Flows Assembly

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Abstract. The paper is devoted to obtaining estimations of the rate of convergence of the intensity of an assembly of Poisson flows to the intensity of a stationary Poisson flow. Analysis of the results shows that this problem should combine analytical and numerical studies. An important role is played by the Central limit theorem for both random variables and stochastic processes which is understood in the sense of C-convergence. Exact asymptotic formulas are derived for intensity of the assembly flow of identical Poisson flows, and estimations of the convergence rate are build for the case of non-identical original flows.

Keywords: Assembly of flows · Asymptotic analysis · Central limit theorem

1 Introduction

In the paper, we analyze the assembly of independent Poisson flows which is interpreted as connection of customers with the same order numbers taken from different flows. The assembly processes may be found in computer networks [1], in conveyor systems for the manufacture of products [2–4], in open queueing networks with a single input flow, division and merging of customers and with sufficiently general configuration of network [5,6], in closed queueing networks with discrete time transitions of batches of customers and dynamic control of service rates [7]. However, the study of the flow of customers coming out after the assembly is a very complicated problem.

It is shown in [8] that the average intensity of the assembled flow tends to the lower of the original Poisson flow intensities while time tends to infinity. However, computational experiments performed by approximating the Poisson distribution with a large parameter by a normal distribution showed that it is possible to improve the obtained estimations of the convergence rate. This paper is devoted to obtaining, in a certain sense, unimproved estimations of the convergence rate of the intensity of the assembly flow to the intensity of a stationary Poisson flow. Analysis of the results shows that this problem should

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combine analytical and numerical methods comparing their results with each other. Moreover, an important role is played by the Central limit theorem for both random variables and stochastic processes which is understood in the sense of C-convergence [9].

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2 Mathematical Model

Assume that there are r independent stationary Poisson processes (we will call them as "original flows") with intensities $\lambda_1, \ldots, \lambda_r$. Let us denote an instants of arrivals in the flows by $t_{k,i}$, where k is the number of original flow and i is the order number of the arrival in this flow. The original flows we denote as $T_k = \{0 \le t_{k,1} \le t_{k,2} \le \ldots\}$, where $k = 1, 2, \ldots,\ k = 1, \ldots, r$. We will call the flow $A_r = \{0 \le \max(t_{1,1}, \ldots, t_{r,1}) \le \max(t_{1,2}, \ldots, t_{r,2}) \le \ldots\}$ as an *assembly of flows* $T_1, \ldots, T_r$ or as an *assembly flow*.

Denote the number of points in k-th flow in interval $[0, t)$ by $n_k(t)$. Then the number of points in the assembly flow $N_r(t)$ in the interval may be expressed as

$$N_r(t) = \min_{k=1,\ldots,r} n_k(t). \tag{1}$$

Flow A_r is not Poisson, because its increments are not independent due to formula (1).

3 Central Limit Theorem for the Assembly Flow

Suppose that several original flows have minimal intensities: $\lambda = \lambda_1 = \ldots = \lambda_s < \lambda_{s+1} \le \ldots \le \lambda_r$, $s \le r$. Then the following statement can be proved.

Theorem 1. *For any $v \in (-\infty, \infty)$, the following limit relation is true:*

$$\mathrm{P}\left\{\frac{N_r(t) - \lambda t}{\sqrt{\lambda t}} > v\right\} \to \left[\int_v^\infty \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du\right]^s,\ t \to \infty. \tag{2}$$

Proof. From formula (1) and the independence of flows $T_1, \ldots, T_r$, the equality follows

$$\mathrm{P}\{N_r(t) > i\} = \prod_{k=1}^{r} \mathrm{P}\{n_k(t) > i\}. \tag{3}$$

Then due to the Central limit theorem, we derive

$$\mathrm{P}\left\{\frac{n_k(t) - \lambda t}{\sqrt{\lambda t}} > v\right\} \to \int_v^\infty \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du, \quad \text{for } k = 1, \ldots, s. \tag{4}$$

and

$$
\mathrm{P}\left\{\frac{n_k(t)-\lambda t}{\sqrt{\lambda t}} > v\right\} = \mathrm{P}\left\{\frac{n_k(t)-\lambda_k t}{\sqrt{\lambda_k t}} > v\sqrt{\frac{\lambda}{\lambda_k}} - (\lambda_k-\lambda)t\right\} \geq
$$

$$
\mathrm{P}\left\{\frac{n_k(t)-\lambda_k t}{\sqrt{\lambda_k t}} > -(\lambda_k-\lambda)t\right\} \to 1,\ t\to\infty,\ \text{for } k=s+1,\dots,r.
$$

So, we obtain

$$
\mathrm{P}\left\{\frac{n_k(t)-\lambda t}{\sqrt{\lambda t}} > v\right\} \to 1,\ t\to\infty, \quad \text{for } k=s+1,\dots,r. \tag{5}
$$

Using formulas (1), (3)–(5), we derive

$$
\mathrm{P}\left\{\frac{N_r(t)-\lambda t}{\sqrt{\lambda t}} > v\right\} = \mathrm{P}\left\{\frac{\min\limits_{k=1,\dots,r} n_k(t)-\lambda t}{\sqrt{\lambda t}} > v\right\} =
$$

$$
\mathrm{P}\left\{\min_{k=1,\dots,r}\frac{n_k(t)-\lambda t}{\sqrt{\lambda t}} > v\right\} =
$$

$$
\prod_{k=1}^{r}\mathrm{P}\left\{\frac{n_k(t)-\lambda t}{\sqrt{\lambda t}} > v\right\} \to \left[\int_v^\infty \frac{1}{\sqrt{2\pi}}\exp(-u^2/2)du\right]^s,\ t\to\infty. \tag{6}
$$

So, the theorem is proved.

Remark 1. The stochastic process $\frac{n_k(tu)-\lambda tu}{\sqrt{\lambda t}}$ as a function of variable $u \geq 0$ tends to Wiener process $\xi_k(u),\ k=1,\dots,s$ while $t\to\infty$. Since process $n_k(t)$ is a process with independent increments, then the stochastic process $\frac{n_k(tu)-\lambda tu}{\sqrt{\lambda t}}$ is also a process with independent increments. Moreover, while $t\to\infty$, due to the Central limit theorem, the increment of process $\frac{n_k(tu)-\lambda tu}{\sqrt{\lambda t}}$ in interval $[u_1,u_2],\ u_1<u_2$, tends to Gaussian random variable with zero mean and variance equal to u_2-u_1. Therefore, if $t\to\infty$, process $\frac{n_k(tu)-\lambda tu}{\sqrt{\lambda t}}$ converges to Wiener process $w_k(u)$ in the sense of C-convergence [9, Chapter 4, § 3, Theorem 10].

Remark 2. Let $r=s$, then stochastic process $\frac{N_r(tu)-\lambda tu}{\sqrt{\lambda t}}$ for $t\to\infty$ converges to process $\min\limits_{k=1,\dots,r} w_k(u)$ in the sense of C-convergence, where $w_1(u),\dots,w_r(u)$ are independent Wiener processes ($u\geq 0$). This statement follows from formula (9) and Remark 1.

4 Limit Relations for Intensity of Assembly of Independent and Identically Distributed Poisson Flows

Consider Markov process $\{n_1(t), \dots, n_r(t)\}$. A jump of this process in instant t from state $(n_1, \dots, n_i, \dots, n_m)$, where $n_i < \min_{k \neq i} n_k$, to state $(n_1, \dots, n_i + 1, \dots, n_r)$ causes a new point to appear in flow A_r in time moment t. Therefore, instant intensity $\overline{\lambda}(t)$ of assembly flow in this moment satisfies the equality

$$\overline{\lambda}(t) = \lambda \sum_{i=1}^{r} \mathrm{P}\left\{ n_i(t) < \min_{k \neq i} n_k(t) \right\}. \tag{7}$$

Lemma 1. *The following equality is true:*

$$\overline{\lambda}(t) = \lambda(1 - \mathrm{P}\{n_1(t) = \ldots = n_r(t)\}). \tag{8}$$

Proof. Let us denote the following sets of indices:

$$J = \{1, \dots, r\}, \ J_i = J \setminus i, \ i = 1, \dots, r.$$

Then equality (7) can be transformed as follows:

$$\overline{\lambda}(t) = \lambda \mathrm{P}\left\{ \bigcup_{i=1}^{r} (n_i(t) < \min_{k \in J_i} n_k(t)) \right\} = \lambda \left(1 - \mathrm{P}\left\{ \bigcap_{i=1}^{r} (n_i(t) \geq \min_{k \in J_i} n_k(t)) \right\} \right)$$
$$= \lambda \left(1 - \mathrm{P}\left\{ \bigcap_{i=1}^{r} (n_i(t) \geq \min_{k \in J} n_k(t)) \right\} \right) = \lambda(1 - \mathrm{P}\{n_1(t) = \ldots = n_r(t)\}).$$

The lemma is proved.

Let us denote $a = \lambda t$ and

$$p(k, a) = \frac{e^{-a} a^k}{k!}, \ k = 0, 1, \dots,$$
$$f(a) = P(n_1(t) = \ldots = n_r(t)) = \sum_{k=0}^{\infty} p^r(k, a).$$

We will search for approximation $g(a)$ of function $f(a)$ in the form

$$g(a) = \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{(x-a)^2}{2a} \right) \right]^r dx =$$
$$(2\pi a)^{-r/2} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-a)^2}{2a/r} \right) dx =$$
$$(2\pi a)^{-r/2} \sqrt{2\pi a/r} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a/r}} \exp\left(-\frac{(x-a)^2}{2a/r} \right) dx = \frac{1}{\sqrt{r}} (2\pi a)^{(1-r)/2}. \tag{9}$$

Theorem 2. *For $\lambda > 0,\ r = 2$, the following limit ratio is true:*

$$\mathrm{P}\{n_1(t) = n_2(t)\} \sim \left(2\sqrt{\pi a}\right)^{-1} \to 0,\ a \to \infty, \tag{10}$$

and therefore, $\lambda(t) \to \lambda,\ \lambda(t) - \lambda \sim \lambda(2\sqrt{\pi\lambda t})^{-1},\ t \to \infty$.

Proof. Indeed, the following equalities are fulfilled:

$$\mathrm{P}\{n_1(t) = n_2(t)\} = \sum_{k=0}^{\infty} \exp(-2a)\frac{a^{2k}}{(k!)^2} = \exp(-2a)B,\ B = \sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2}.$$

Here $B = B(a)$ is the Infeld function [10, Chapter 4, Sect. 11] satisfying the asymptotic relation

$$B(a) = \frac{\exp(2a)}{2\sqrt{\pi a}}\left(1 + O\left(\frac{1}{a}\right)\right). \tag{11}$$

Replacing here a by λt, we derive relation (10). The theorem is proved.

Theorem 3. *If $\lambda > 0,\ r > 2,\ \frac{1}{2} < \gamma < \frac{2}{3}$, then the following limit relation takes place:*

$$f(a) = g(a)(1 + O(a^{3\gamma-2})) \sim g(a),\ a \to \infty, \tag{12}$$

and therefore, $\lambda(t) \to \lambda,\ \lambda(t) - \lambda \sim \lambda\dfrac{(2\pi\lambda t)^{(1-r)/2}}{\sqrt{r}},\ t \to \infty$.

Proof. Consider the following integrals:

$$g_1(a) = \int_{-\infty}^{a-a^\gamma} \left[\frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{(x-a)^2}{2a}\right)\right]^r dx,$$
$$g_2(a) = \int_{a+a^\gamma}^{\infty} \left[\frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{(x-a)^2}{2a}\right)\right]^r dx,$$
$$g_3(a) = \int_{a-a^\gamma}^{a+a^\gamma} \left[\frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{(x-a)^2}{2a}\right)\right]^r dx,$$
$$g(a) = g_1(a) + g_2(a) + g_3(a). \tag{13}$$

Let us prove the following supplementary statement.

Lemma 2. *The following limit relations are true:*

$$g_1(a) = g_2(a) = o(g(a)),\ g_3(a) = g(a)(1 + o(g(a))),\ a \to \infty. \tag{14}$$

Proof. By replacing variable $t = x - a$, we obtain the equalities

$$g_1(a) = \int_{-\infty}^{-a^\gamma} \left[\frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{t^2}{2a}\right)\right]^r dt,\ g_2(a) = \int_{a^\gamma}^{\infty} \left[\frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{t^2}{2a}\right)\right]^r dt.$$

From here, we get the following relations (for $a \to \infty$):

$$g_1(a) = g_2(a) = \int_{a^\gamma}^{\infty} \frac{(2\pi a)^{-r/2}}{a^{-1} r t} \exp\left(-\frac{rt^2}{2a}\right) d\left(\frac{rt^2}{2a}\right) \leq$$

$$\frac{(2\pi a)^{-r/2}}{r a^{\gamma-1}} \int_{a^\gamma}^{\infty} \exp\left(-\frac{rt^2}{2a}\right) d\frac{rt^2}{2a} \leq \frac{(2\pi a)^{-r/2}}{r a^{\gamma-1}} \exp\left(-\frac{r a^{2\gamma-1}}{2}\right) = o(g(a)).$$

From these relations and formulas (9), (13), limit relations (14) follow. The lemma is proved.

Let us now consider the sums

$$f_1(a) = \sum_{0 \leq k < a - a^\gamma} \left(\frac{e^{-a} a^k}{k!}\right)^r, \tag{15}$$

$$f_2(a) = \sum_{a + a^\gamma < k \leq \infty} \left(\frac{e^{-a} a^k}{k!}\right)^r, \; f_3(a) = \sum_{a - a^\gamma \leq k \leq a + a^\gamma} \left(\frac{e^{-a} a^k}{k!}\right)^r. \tag{16}$$

Further, we denote an integer part of some real number x by $[x]$.

Let us prove an additional supplementary statement.

Lemma 3. *The following limit relations are true:*

$$f_1(a) = O\left(\frac{a}{(2\pi a)^{r/2}} \exp\left(-\frac{r a^{2\gamma-1}}{2}\right)\right) = o(g(a)), \quad a \to \infty, \tag{17}$$

$$f_2(a) = O\left(\frac{a}{(2\pi a)^{r/2}} \exp\left(-\frac{r a^{2\gamma-1}}{2}\right)\right) = o(g(a)), \quad a \to \infty. \tag{18}$$

Proof. We construct an estimation of $f_1(a)$, assuming $c = [a - a^\gamma] \sim a, \; a \to \infty$:

$$f_1(a) \leq c \left(\frac{e^{-a} a^c}{c!}\right)^r \sim a \left(\frac{e^{-a} a^c}{c^c e^{-c} \sqrt{2\pi a}}\right)^r \leq$$

$$\frac{a}{(2\pi a)^{r/2}} \left(\frac{e^{-a} a^{a - a^\gamma}}{(a - a^\gamma - 1)^{a - a^\gamma - 1} e^{-a + a^\gamma}}\right)^r = \frac{a}{(2\pi a)^{r/2}} e^{r F_1(a)}, \tag{19}$$

where

$$F_1(a) = -a^\gamma + (a - a^\gamma) \ln a - (a - a^\gamma - 1) \ln(a - a^\gamma - 1) = -\frac{a^{2\gamma-1}}{2}(1 + o(1)). \tag{20}$$

Thus, from the condition $\frac{1}{2} < \gamma$, definition of function $g(a)$ and formulas (15), (19), (20), it leads us to (17).

We construct an estimation of $f_2(a)$, assuming in the proof of Lemma 3 that $d = [a + a^\gamma] \sim a, \; a \to \infty$:

$$f_2(a) \leq \sum_{d \leq k} \left(e^{-a} \frac{a^k}{k!}\right)^r \leq \left(\frac{e^{-a} a^d}{d!}\right)^r \sum_{k \geq 0} \left(\frac{a}{d}\right)^{kr} \sim \left(\frac{e^{-a} a^d}{d!}\right)^r \frac{a^{1-\gamma}}{r}. \tag{21}$$

So, for $a \to \infty$ we derive

$$\left(\frac{e^{-a}a^d}{d!}\right)^r \sim \left(\frac{e^{-a}a^d}{d^d e^{-d}\sqrt{2\pi a}}\right)^r \leq$$

$$\left(\frac{e^{-a}a^{a+a^\gamma}}{(a+a^\gamma-1)^{a+a^\gamma-1}e^{-a-a^\gamma}\sqrt{2\pi a}}\right)^r = \frac{1}{(2\pi a)^{r/2}}e^{rF_2(a)}, \tag{22}$$

where

$$F_2(a) = a^\gamma + (a+a^\gamma)\ln a - (a+a^\gamma-1)\ln(a+a^\gamma-1) = -\frac{a^{2\gamma-1}}{2}(1+o(1)). \tag{23}$$

From (16), (21)–(23) and the condition $\frac{1}{2} < \gamma$, we obtain (18). The lemma is proved.

Let us denote

$$\varphi_3(a) = \sum_{a-a^\gamma \leq k \leq a+a^\gamma} (2\pi a)^{-r/2} \exp\left(-\frac{r(k-a)^2}{2}\right)$$

and prove the following supplementary statements.

Lemma 4. *The following limit relation is true:*

$$f_3(a) = \varphi_3(a)(1+O(a^{3\gamma-2})),\ a \to \infty. \tag{24}$$

Proof. We analyze the expression $e^{-ra}\dfrac{a^{rk}}{(k!)^r}$ using the Stirling formula in the form

$$k! = k^k e^{-k}\sqrt{2\pi k}\exp\left(\frac{\theta(k)}{12k}\right),\ 0 \leq \theta(k) \leq 1.$$

From this formula, it follows that

$$(k!)^{-r} = k^{-rk}e^{rk}(2\pi k)^{-r/2}\exp\left(-\frac{r\theta(k)}{12k}\right),\ 0 \leq \theta(k) \leq 1. \tag{25}$$

It is obvious that the following relations are true:

$$\sup_{k:\ |k-a| \leq a^\gamma} \left|\exp\left(-\frac{r\theta(k)}{12k}\right) - 1\right| = O(a^{-1}), \tag{26}$$

$$\sup_{k:\ |k-a| \leq a^\gamma} \left|\frac{(2\pi k)^{-r/2}}{(2\pi a)^{-r/2}} - 1\right| = O(a^{\gamma-1}). \tag{27}$$

From formulas (25)–(27), we obtain the following relation:

$$\sup_{k:\ |k-a| \leq a^\gamma} \left|e^{-ra}\frac{a^{rk}}{(k!)^r} \cdot e^{ra}\left(\frac{ea}{k}\right)^{-rk}(2\pi a)^{r/2} - 1\right| = O(a^{\gamma-1}).$$

The notation

$$p(k,a) = q(k,a)(1+O(a^t)),\ |k-a| \le a^\gamma,\ a \to \infty$$

means that the following relation is satisfied:

$$\sup_{k:\ |k-a|\le a^\gamma} \left| \frac{p(k,a)}{q(k,a)} - 1 \right| = O(a^t),\ a \to \infty.$$

Therefore, the following relation is true:

$$e^{-ra}\frac{a^{rk}}{(k!)^r} = e^{-ra}\left(\frac{ea}{k}\right)^{rk}(2\pi a)^{-r/2}(1+O(a^{\gamma-1})),\ |k-a| \le a^\gamma,\ a \to \infty. \quad (28)$$

Using the Taylor series expansion of the function $\ln(1+u) = u - \frac{u^2}{2} + O(u^3),\ |u| < 1$, we evaluate $\ln\left[e^{-a}\left(\frac{ea}{k}\right)^k\right]$. To do this, we set $k = a+v,\ |v| \le a^\gamma$ and evaluate the ratio

$$\ln\left[e^{-a}\left(\frac{ea}{k}\right)^k\right] = -a + (a+v)(1+\ln a - \ln a - \ln(1+v/a)) = -\frac{v^2}{2a} + O\left(\frac{v^3}{a^2}\right),$$

from which it follows that

$$e^{-a}\left(\frac{ea}{k}\right)^k = \exp\left(-\frac{(k-a)^2}{2a}\right)\left(1+O\left(a^{3\gamma-2}\right)\right),\ |k-a| \le a^\gamma,\ a \to \infty. \quad (29)$$

From expressions (28) and (29), we obtain the asymptotic relation

$$e^{-ra}\frac{a^{rk}}{(k!)^r} = (2\pi a)^{-r/2}\exp\left(-\frac{r(k-a)^2}{2a}\right)(1+O(a^{3\gamma-2})),\ |k-a| \le a^\gamma,\ a \to \infty.$$

By combining this relation with formula (16), we obtain limit relation (24). The lemma is proved.

Corollary 1. *It follows from Lemma 4 that the following asymptotic formula holds uniformly for all $k:\ |k-a| \le a^\gamma$:*

$$p(k,a) \sim \exp\left(-\frac{(k-a)^2}{2a}\right)\frac{1}{\sqrt{2\pi a}},\ a \to \infty.$$

Lemma 5. *The following limit ratio is true:*

$$\varphi_3(a) = g_3(a)(1+O(a^{\gamma-1})),\ a \to \infty. \quad (30)$$

Proof. Without a significant generality constraint (to simplify the proof), we assume that $a,\ a^\gamma$ are integer. Then the following equality is true:

$$g_3(a) = \sum_{a-a^\gamma \le k < a+a^\gamma} (2\pi a)^{-r/2}\int_k^{k+1}\exp\left(-\frac{r(x-a)^2}{2a}\right)dx.$$

For $k < a$, the function $\exp\left(-\frac{r(x-a)^2}{2a}\right)$ is monotonically increasing, and for $k \geq a$, it decreases monotonically on the interval $[k, k+1]$. So, the following relation is true:

$$\exp\left(-\frac{r(k-a)^2}{2a}\right) = \exp\left(-\frac{r(k+1-a)^2}{2a}\right)(1+O(a^{\gamma-1})),\ |k-a| \leq a^\gamma,\ a \to \infty.$$

It follows that the following relation is satisfied:

$$\exp\left(-\frac{r(k-a)^2}{2a}\right) = \int_k^{k+1} \exp\left(-\frac{r(x-a)^2}{2a}\right) dx(1+O(a^{\gamma-1})),\ |k-a| \leq a^\gamma,$$

when $a \to \infty$, and in addition, we have $\varphi_3(a+a^\gamma, a) = (2\pi a)^{-r/2}\exp(-ra^{2\gamma-1})$. From these relations and formulas (12), (14), we derive (30). The lemma is proved.

From formulas (24), (30), we obtain the relation

$$f_3(a) = g_3(a)(1+O(a^{3\gamma-2}))(1+O(a^{\gamma-1})) = g_3(a)(1+O(a^{3\gamma-2})). \tag{31}$$

Combining Formulas (9), (14), (17), (18), and (31), we obtain (12). So, Theorem 3 is proved.

Remark 3. The series $\sum_{k\geq 0}\left(\frac{a^k}{k!}\right)^r$ considered in Theorem 3 is a generalized hypergeometric series. However, it is impossible to use a well-known asymptotic formulas [11, Chapter 16] for it.

Remark 4. We present results of a computational experiment illustrating the accuracy of the obtained approximations. Denote error of the approximation by $\Delta(a) = \left|\frac{f(a)-g(a)}{f(a)}\right|$. Values of $\Delta(a)$ are presented in Table 1. We may notice that the error is decreasing while a grows.

Table 1. Values of $\Delta(a)$ for $r = 2, 5, 20;\ a = 10^k,\ k = 1, \ldots, 6$.

r	a					
	10	10^2	10^3	10^4	10^5	10^6
2	6.4×10^{-3}	6.3×10^{-4}	6.3×10^{-5}	6.3×10^{-6}	6.2×10^{-7}	6.2×10^{-8}
5	2.0×10^{-2}	2.0×10^{-3}	2.0×10^{-4}	2.0×10^{-5}	2.0×10^{-6}	2.0×10^{-7}
20	8.3×10^{-2}	8.3×10^{-3}	8.3×10^{-4}	8.3×10^{-5}	8.3×10^{-6}	8.2×10^{-7}

5 Assembly of Independent Flows with Different Intensities

Consider now the case when there are two Poisson flows $T_1,\ T_2$ with intensities $\lambda_1, \lambda_2 :\ \lambda_1 < \lambda_2$, and denote $d = \lambda_2 t,\ cd = \lambda_1 t$, so,

$$0 < c = \frac{\lambda_1}{\lambda_2} < 1.$$

For instant intensity of assembly flow $\overline{\lambda}(t)$ in this case, we have

$$\begin{aligned}\overline{\lambda}(t) &= \lambda_1 P(n_2(t) > n_1(t)) + \lambda_2 P(n_1(t) > n_2(t)) = \lambda_1(P(n_1(t) > n_2(t))\\ &+P(n_2(t) > n_1(t))) + (\lambda_2 - \lambda_1)P(n_1(t) > n_2(t)) = \lambda_1(1 - P(n_1(t) = n_2(t)))\\ &+(\lambda_2 - \lambda_1)P(n_1(t) > n_2(t)) = \lambda_1 - \lambda_1 P(n_1(t) \geq n_2(t)) + \lambda_2 P(n_1(t) > n_2(t)),\end{aligned}$$

therefore,

$$|\overline{\lambda}(t) - \lambda_1| \leq \lambda_2 P(n_1(t) \geq n_2(t)), \tag{32}$$

where

$$P(n_1(t) \geq n_2(t)) = \sum_{k=0}^{\infty} e^{-d}\frac{d^k}{k!}\sum_{i=k}^{\infty} e^{-cd}\frac{(cd)^i}{i!} = G(d). \tag{33}$$

Consider function $G(d)$ in a form of the sum $G(d) = G_1(d) + G_2(d)$, where

$$G_1(d) = \sum_{k>d} e^{-d}\frac{d^k}{k!}\sum_{i=k}^{\infty} e^{-cd}\frac{(cd)^i}{i!}, \quad G_2(d) = \sum_{k\leq d} e^{-d}\frac{d^k}{k!}\sum_{i=k}^{\infty} e^{-cd}\frac{(cd)^i}{i!}.$$

For a fixed $c :\ 0 < c < 1$, we define the function $\psi(c) = c - 1 - \ln c$. Function $\psi(c)$ satisfies the relations $\psi(1) = 0,\ \psi'(c) = 1 - 1/c < 0$, so, $\psi(c)$ is positive and monotonically decreasing for $0 < c < 1$.

We will call that positive functions $p(d)$ and $q(d)$ satisfy the relation

$$p(d) \preceq q(d),\ d \to \infty, \ \text{ if } \ \limsup_{d\to\infty}\frac{p(d)}{q(d)} < \infty.$$

Lemma 6. *For any $c :\ 0 < c < 1$, the following formula holds:*

$$d^{-1/2}\exp(-d\psi(c)) \preceq G_1(d) \preceq d^{1/2}\exp(-d\psi(c)),\ d \to \infty. \tag{34}$$

Proof. Really, we have:

$$\begin{aligned}G_1(d) &\leq \sum_{k>d} e^{-d}\frac{d^k}{k!}\sum_{i=[d]}^{\infty} e^{-cd}\frac{(cd)^i}{i!} \leq \sum_{i=[d]}^{\infty} e^{-cd}\frac{(cd)^i}{i!} \leq\\ &\frac{e^{-cd}(cd)^d}{[d]!}\sum_{i=[d]}^{\infty}\left(\frac{cd}{[d]}\right)^{i-[d]} = \frac{e^{-cd}(cd)^d}{[d]!}\left(1 - \frac{cd}{[d]}\right)^{-1} \sim \frac{e^{-cd}(cd)^d}{(1-c)[d]!}.\end{aligned}$$

Due to the Stirling formula, we derive

$$\frac{e^{-cd}(cd)^d}{[d]!} \leq \frac{e^{-cd}(cd)^d}{[d]^{[d]}e^{-[d]}\sqrt{2\pi[d]}} \leq \frac{e^{d-cd}(cd)^d}{[d]^d\sqrt{2\pi[d]}} \leq$$

$$\frac{e^{d-cd}(cd)^d}{(d-1)^{d-1}\sqrt{2\pi(d-1)}} \sim e\sqrt{\frac{d}{2\pi}}\exp(-d\psi(c)).$$

As a result, we come to the right relation in formula (34).

We now construct the lower bound of the function $G_1(d)$, assuming $j = [d]+1,\ d \to \infty$:

$$e^{-d}\frac{d^j}{j!} \geq e^{-1/12d}e^{-d+j}\frac{[d]^j}{j^j\sqrt{2\pi j}} \geq \frac{1}{\sqrt{2\pi j}}\left(1+\frac{1}{[d]}\right)^{-j} \sim \frac{1}{e\sqrt{2\pi d}}, \tag{35}$$

$$e^{-cd}\frac{(cd)^j}{j!} \sim e^{-cd}\frac{(cd)^j}{j^je^{-j}\sqrt{2\pi j}} \geq e^{-cd+j}\frac{c^{d+1}d^j}{j^j\sqrt{2\pi j}} \sim \frac{c\exp(-d\psi(c))}{e\sqrt{2\pi d}}. \tag{36}$$

From formulas (35) and (36), the left relation in formula (34) follows. The lemma is proved.

Let us fix $s:\ 0 < c < s < 1$, let $l = [sd]$, and estimate $G_2(d) = G_2'(d)+G_2''(d)$, where

$$G_2'(d) = \sum_{0\leq k<sd} e^{-d}\frac{d^k}{k!}\sum_{i=k}^{\infty} e^{-cd}\frac{(cd)^i}{i!},\ G_2''(d) = \sum_{sd\leq k\leq d} e^{-d}\frac{d^k}{k!}\sum_{i=k}^{\infty} e^{-cd}\frac{(cd)^i}{i!}.$$

Denote $\mu(c,s) = c - s(1+\ln c - \ln s),\ q(s) = 1 - s(1-\ln s),\ 0 < c < s < 1$. For any $c, s:\ 0 < c < s < 1$, the relations

$$\mu(c,1) = \psi(c) > 0,\ \frac{\partial\mu(c,s)}{\partial s} = -\ln\frac{c}{s} > 0$$

take place for fixed c. Function $\mu(c,s)$ increases on argument $s:\ c < s < 1$ and

$$\mu(c,c) = 0,\ \mu(c,s) > 0,\ c < s < 1.$$

Function $q(s)$ satisfies the relations

$$q(1) = 0,\ q'(s) = \ln s < 0,\ q(s) > 0,\ 0 < s < 1,$$

and hence, it is positive and monotonically decreasing for $0 < s < 1$.

Lemma 7. *For any $c,\ s:\ 0 < c < s < 1$, the following formula holds:*

$$d^{-1/2}\exp(-dq(s)) \preceq G_2'(d) \preceq d^{1/2}\exp(-dq(s)),\ d \to \infty. \tag{37}$$

Proof. Function $G_2'(d)$ satisfies the following relations for $cd > 2$:

$$G_2'(d) \leq l\frac{e^{-d}d^l}{l!} \sim e^{-d+l}\sqrt{\frac{l}{2\pi}}\left(\frac{d}{l}\right)^l \leq e^{-d+sd}\sqrt{\frac{l}{2\pi}}\left(\frac{d}{sd-1}\right)^{sd} \sim$$

$$\sqrt{\frac{sd}{2\pi}}\frac{l\cdot d^l}{l^l\sqrt{2\pi sd}} \sim \frac{e}{\sqrt{2\pi sd}}\exp(-dq(s)).$$

Thus, the right relation in formula (37) is true.

On the other hand $G_2'(d) \geq \frac{e^{-d}d^l}{l!}\left(1 - \sum\limits_{0\leq i\leq l} e^{-cd}\frac{(cd)^i}{i!}\right)$, where

$$\frac{e^{-d}d^l}{l!} \sim \frac{e^{-d+l}d^l}{l^l\sqrt{2\pi l}} \geq \frac{e^{-d+l}}{\sqrt{2\pi l}}\left(\frac{1}{s}\right)^l \sim \frac{\exp(-dq(s))}{e\sqrt{2\pi sd}}, \tag{38}$$

and

$$\sum_{0\leq i<l}\frac{e^{-cd}(cd)^i}{i!} \leq sde^{-cd}\frac{(cd)^l}{l!} \sim \sqrt{\frac{1}{2\pi sd}}e\exp(-d\mu(s,c)) \to 0,\ d\to\infty. \tag{39}$$

From (38), (39), we derive the left relation of (37). The lemma is proved.

Lemma 8. *For any* $c,\ s:\ 0<c<s<1$*, the following formula holds:*

$$d^{-1}\exp(-d(\mu(c,s)+q(s)) \preceq G_2''(d) \preceq d^{1/2}\exp(-d\mu(c,s))),\ d\to\infty. \tag{40}$$

Proof. Function $G_2''(d)$ satisfies the following relations for $cd > 2$:

$$G_2''(d) \leq \sum_{i\geq sd} e^{-cd}\frac{(cd)^i}{i!} \leq e^{-cd}\frac{(cd)^l}{l!}\sum_{i\geq 0}\frac{(cd)^i}{l^i} =$$

$$e^{-cd}\frac{(cd)^l}{l!}\left(1-\frac{cd}{l}\right)^{-1} \sim e^{-cd}\frac{(cd)^l}{l!}\left(1-\frac{c}{s}\right)^{-1}.$$

We derive

$$e^{-cd}\frac{(cd)^l}{l!} \sim e^{-cd+l}\frac{(cd)^{sd}}{l^l\sqrt{2\pi sd}} \leq e^{-cd+sd}\frac{(cd)^{sd}}{(sd-1)^{sd-1}\sqrt{2\pi sd}} \leq$$

$$sde^{-cd+sd}\frac{(cd)^{sd}}{(sd-1)^{sd}\sqrt{2\pi sd}} = \sqrt{\frac{sd}{2\pi}}e^{-cd+sd}\left(\frac{c}{s}\right)^{sd}\left(1-\frac{1}{sd}\right)^{-sd} \sim$$

$$e\sqrt{\frac{sd}{2\pi}}\exp(-d\mu(c,s)). \tag{41}$$

Hence, the right relation in formula (40) is true.

At the same time, using (38) in a similar way to formula (41), we obtain

$$G_2''(d) \geq \frac{e^{-d}d^l}{l!}\cdot\frac{e^{-cd}(cd)^l}{l!} \sim \frac{\exp(-dq(s))}{e\sqrt{2\pi sd}}\cdot\frac{\exp(-d\mu(c,s))}{e\sqrt{2\pi sd}}. \tag{42}$$

From (47), the left relation of formula (40) follows. The lemma is proved.

For fixed $c, s:\ 0 < c < s < 1$, we define the functions

$$\vartheta(c,s) = \min(\mu(c,s), q(s));\ \nu(c,s) = \min(\vartheta(c,s), \psi(c)),$$

$$\alpha(c) = \sup_{s:\ c<s<1} \nu(c,s),\ s^*(c) = -\tfrac{1-c}{\ln c}.$$

Lemma 9. *For any $c:\ 0 < c < 1$, the following formula holds:*

$$\alpha(c) = q(s^*(c)). \tag{43}$$

Proof. Since function $\mu(c,s)$ is increasing, the function $q(s)$ is decreasing for $s:\ \ c \le s \le 1$, and $\mu(c,c) = 0,\ q(1) = 0,\ s = 1$, then there exists the unique point $s^*(c) = -\dfrac{1-c}{\ln c} > c$ that satisfies the equality $\mu(c, s^*(c)) = q(s^*(c))$. Therefore, the following equality is fulfilled:

$$\sup_{s:\ c<s<1} \vartheta(c,s) = q(s^*(c)). \tag{44}$$

Now we prove that the inequality $q(s^*(c)) < \psi(c)$ holds. Indeed, since $c < s^*(c)$ and function $q(s)$ is decreasing, then $q(s^*(c)) < q(c)$. Therefore, the function

$$\omega(c) = q(c) - \psi(c) = 2 - 2c + (1+c)\ln c$$

satisfies the equality $\omega(1) = 0$, and its derivative satisfies $\omega'(c) = \dfrac{q(c)}{c} > 0$. This means that $\omega(c) < 0$ for $0 < c < 1$. So, the following inequalities are true:

$$q(s^*(c)) < q(c) < \psi(c),\ 0 < c < s^*(c) < 1. \tag{45}$$

From relations (44), (45), we obtain formula (43) for a fixed $c:\ 0 < c < 1$. The lemma is proved.

Theorem 4. *For any $c:\ 0 < c < 1$, the following formula holds:*

$$d^{-1}\exp(-d\alpha(c)) \preceq G(d) \le d^{1/2}\exp(-d\alpha(c)), \tag{46}$$

and therefore, $\lambda(t) \to \lambda,\ \lambda(t) - \lambda = G(\lambda_2 t)$ while $t \to \infty$.

Proof. For any $c, s:\ 0 < c < s < 1$, it follows from Lemmas 6–9 that

$$G(d) \preceq d^{1/2}\exp(-d\min(\psi(c), q(s), \mu(c,s)).$$

So, for any $c:\ 0 < c < 1$, we derive

$$G(d) \preceq d^{1/2}\exp(-dq(s^*(c)) = d^{1/2}\exp(-d\alpha(c)).$$

The right relation in (46) is proved.

For $c, s:\ 0 < c < s < 1$, it follows from Lemmas 6–9 that

$$G(d) \succeq d^{-1}\exp(-d\min(\psi(c), q(s), \mu(c,s) + q(s)).$$

Assuming $s = s^*(c)$ in the last inequality, we obtain

$$G(d) \succeq d^{-1}\exp(-d\min(\psi(c), q(s^*(c)), \mu(c, s^*(c)) + q(s^*(c)))$$

$$= d^{-1}\exp(-d\min(\psi(c), q(s^*(c))) = d^{-1}\exp(-d\alpha(c).$$

The left relation in formula (46) is proved. So, Theorem 4 is proved.

Remark 5. In Remark 4, the estimation of probability $\mathrm{P}\{n_1(t) = \ldots = n_r(t)\}$ uses a Gaussian approximation of a Poisson distribution with a large parameter. It is shown that this approximation gives results similar to the results of the analytical study. Consider how this approximation works when estimating the probability $\mathrm{P}\{n_1(t) \geq n_2(t)\}$.

To do this, we write the following approximations of random variables $n_1(t)$ and $n_2(t)$:

$$n_1(t) \approx \sqrt{cd}\xi_1 + cd, \quad n_2(t) \approx \sqrt{d}\xi_2 + d,$$

where ξ_1 and ξ_2 are independent random variables having a standard normal distribution (with zero mean and variance equal to one). Then by analogy with the proof of Lemma 5, we can construct a Gaussian approximation of the probability

$$\mathrm{P}\{n_1(t) \geq n_2(t)\} \approx \mathrm{P}\{\sqrt{cd}\,\xi_1 + cd \geq \sqrt{d}\,\xi_2 + d\} =$$

$$\mathrm{P}\{\xi_2 \leq \sqrt{c}\,\xi_1 + \sqrt{d}(c-1)\} = S(d),\ d \to \infty.$$

Denote a random variable with a standard normal distribution by η and put $h = (c-1)\sqrt{\dfrac{d}{c+1}}$. Since random vector (ξ_1, ξ_2) has a two-dimensional normal distribution with zero mean and with an identity covariance matrix, then using well-known asymptotic formula

$$\mathrm{P}\{\eta > R\} \sim \frac{1}{R\sqrt{2\pi}}\exp\left(-\frac{R^2}{2}\right),\ R \to \infty,$$

it is possible to obtain the following ratio based on the Gaussian approximation:

$$\mathrm{P}\{n_1(t) \geq n_2(t)\} \approx \frac{1}{h\sqrt{2\pi}}\exp\left(-\frac{h^2}{2}\right) = \frac{\sqrt{c+1}}{\sqrt{2\pi d}(c-1)}\exp\left(-d\cdot\frac{(c-1)^2}{2(c+1)}\right)$$

$$= \frac{\sqrt{c+1}}{\sqrt{2\pi d}(c-1)}\exp(-dA(c)) = S(d),\ A(c) = \frac{(c-1)^2}{2(c+1)},\ d \to \infty. \quad (47)$$

Now compare factors $\alpha(c)$ and $A(c)$ in the exponents of (46) and (47). When $c = 5/6$, we have $\alpha(c) \approx 0,0038,,\ A(c) \approx 0,0076$. If $c = 2/3$, then $\alpha(c) \approx 0,0168,\ A(c) \approx 0,0333$. Thus, factor $A(c)$ calculated by the Gaussian approximation is greater than factor $\alpha(c)$ calculated analytically.

We denote $\delta(d) = \left|\dfrac{G(d) - S(d)}{G(d)}\right|$ and numerically evaluate an accuracy of the Gaussian approximation for $c = 5/6$ and $c = 2/3$. The results are presented in Tables 2 and 3. We may notice that the rate of decreasing of values $\delta(d)$ decreases while d grows for the case $c = 5/6$ (Table 2). On other hand, after some decreasing, value of $\delta(d)$ starts to grow while d grows for the case $c = 2/3$ (Table 3). Thus, the results given in Tables 2 and 3 indicate a much worse quality of the Gaussian approximation than the results given in Table 1.

Table 2. Values of $\delta(d)$ for $c = 5/6$.

d	100	200	500	1000	2000
$\delta(d)$	0.267	0.143	0.051	0.021	0.018

Table 3. Values of $\delta(d)$ for $c = 2/3$.

d	10	50	100	200	500
$\delta(d)$	0.321	0.059	0.029	0.047	0.192

Remark 6. Using the proof of Lemma 1, it is easy to consider the case of assembling r independent Poisson flows with intensities $\lambda_1 = \lambda_2 = \ldots = \lambda_s < \lambda_{s+1} \leq \ldots \leq \lambda_r$, to get the inequality

$$|\overline{\lambda}(t) - \lambda_1| \leq \sum_{i=s+1}^{r} \lambda_i P(n_1(t) \geq n_i(t))$$

and to use Theorem 4 for estimating the probabilities $\mathrm{P}\{n_1(1) \geq n_i(t)\},\ i = s+1, \ldots, r$. The results of performed numerical experiments are very sensitive to the correct or incorrect choice of the corresponding asymptotic formulas.

6 Convergence of Assembly Flow A_2 to Poisson Flow

Consider the union T^2 of independent Poisson flows T_1 and T_2 with equal intensities λ. It is well-known that T^2 is a Poisson flow with intensity 2λ. Denote its points as $T^2 = \{0 = t(0) < t(1) < \ldots\}$.

Also, consider assembly A_2 of the flows T_1 and T_2. Its points $\{0 = t_0 < t_1 < t_2 < \ldots\}$ are defined by the expressions

$$t_k = \inf\{t > t_{k-1} : \ n_1(t) = n_2(t)\}. \tag{48}$$

Define the Markov process $\nu(t) = n_2(t) - n_1(t)$ with state space $\{0, \pm1, \pm2, \ldots\}$ and transient intensities $\lambda_{k,k+1} = \lambda_{k,k-1} = \lambda,\ k = 0, \pm1, \pm2, \ldots$

Process $\nu(t)$ has jumps ± 1 in the points of flow T^2, and its zeroing points coincide with points $0 = t_0 < t_1 < t_2 < \dots$ of assembly flow A_2. The random sequence $\{\nu(t(j)),\ j = 0, 1, 2, \dots\}$ is a symmetric random walk on the set $\{0, \pm 1, \pm 2, \dots\}$, therefore, accordingly to [12, Chapter III, § 3, Lemma 1], we can write

$$\mathrm{P}\{\nu(t(2j)) = 0\} = C_{2j}^{j} 2^{-2j} = p_{2j} \leq p_{2(j+1)},\ j = 1, 2, \dots,\ p_{2j} \sim \frac{1}{\sqrt{\pi j}},\ j \to \infty. \tag{49}$$

Define the random event

$$\bigcup_{j=k}^{k+K} \{\nu(t(2j)) = 0\} = \bigcup_{j=2k}^{2(k+K)} \{\nu(t(j)) = 0\}\,.$$

Using (49) for the given ε and K, we can derive the following expression:

$$k(\varepsilon, K) = \left[\frac{K^2}{\pi \varepsilon^2}\right], \tag{50}$$

and for any $k > k(\varepsilon, K)$ we obtain

$$P\left\{\bigcup_{j=2k}^{2(k+K)} \{\nu(t(j)) = 0\}\right\} \leq \frac{K}{\sqrt{\pi k(\varepsilon, K)}} \leq \varepsilon. \tag{51}$$

Then due to (51), the equality $n_1(t) = n_2(t)$ does not hold in any $2K$ points following the moment $t(2k(\varepsilon, K))$. Therefore, at this time interval, the assembly flow is Poisson with parameter λ with probability not greater than ε. Note that in this case, due to formula (50), value of $k(\varepsilon, K)$ increases quite rapidly while ε decreases.

7 Conclusion

Despite the apparent simplicity of the considered model of the assembly flow of independent Poisson flows, the study have shown that the model is quite complex for the analysis. In the paper, we have obtained various versions of the Central limit theorem for the assembly flow both in terms of random variables and in terms of stochastic processes. Exact asymptotic formulas are derived for intensity of the assembly flow of identical Poisson flows, and estimations of the convergence rate are build for the case of non-identical original flows. Estimations of the convergence rate of the assembly flow of identical Poisson flows to a Poisson flow are derived.

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