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ATOMS, MOLECULES, OPTICS

Semiclassical Description of Undulator Radiation

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Abstract—We present a semiclassical approximation for treating the radiation from classical currents. In particular, we present exact quantum states of the quantized electromagnetic field interacting with classical currents. These states are used to calculate a probability of many-photon radiation from the vacuum initial state of the electromagnetic field. In this manner, in the present article, we study characteristics of electromagnetic radiation of a planar undulator. We find the total radiated energy and its spectral-angular distribution. We compare our results with ones obtained in the framework of classical electrodynamics, discussing differences introduced by accurate accounting for the quantum nature of electromagnetic radiation and present results of some numerical calculations that confirm, in particular, the latter discussion. In Appendix we present the calculation of the radiated energy using an alternative parametrization of the trajectory of electrons moving in a planar undulator.

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1. INTRODUCTION

In the general case, accelerated charged particles emit an electromagnetic radiation. For example, synchrotron (SR) radiation accompanies a motion of charged particles along circular trajectories, which can be consequence of their motion in an external magnetic field [1]. SR has many important applications in physics, medicine and industry (see, for example, [2]). Ginzburg first proposed a practical use of rapidly moving charged particles as a radiation source [3], see also [4]. Another source of radiation closely related to SR are periodic magnetic structures called undulators or wigglers. The original term “undulator” was introduced by Motz [5], who proposed several applications for such a radiation source, namely production of energy in rather specific spectral bands (millimeter to infrared radiation), speed monitoring for electron beams produced by accelerating devices, speed measurement for fast individual electrons or other particles, including mesons or protons.

In the framework of classical electrodynamics a formula for the angular distribution of the SR power was obtained by Schott [6]. An alternative derivation of this result and its deep analysis, especially for high-energy relativistic electrons, was given by Schwinger [7]. The essence of quantum corrections to the classical result was first pointed out in [8]. Consistent calculation of such corrections first appeared in the works [9] using Furry picture [10] (or exact solutions of Dirac equations with a magnetic field). Using his source theory [11] Schwinger presented an original derivation of similar results [12]. Recently, a description of SR that uses a new intermediate approach allowing one to consider electron current classically while taking quantum nature of electromagnetic field into account exactly was given in [13].

Until now, most of the works on the emission of electrons moving in undulators were carried out using methods of classical electrodynamics. In this approach, the undulator radiation (UR) is calculated using the Lienard-Wiechert potentials for a moving high-energy electron. Thus classical expressions for electric and magnetic fields are calculated and the energy flux, intensity or power are found through the Umov–Poynting vector. In 1951, Motz [5] had presented a planar undulator scheme and studied the corresponding radiation from the electrons moving in such a device. Important contribution to the UR problem was done in the works by Alferov et al. [14]. In particular, the authors calculated radiation in a spiral undulator, presented a spectral-angular distribution of the radiation intensity in various approximations. Radiation of relativistic electrons, moving in an undulator, in particular, in the finite length device, was studied in [15, 16]. The results obtained have often been interpreted in terms of the photon emission; for example, in the works [5, 14]. In their work [17] Baier et al., using the Schwinger method, and some adequate approximations, studied spectral-angular distribution of the radiation intensity of electrons moving in periodic magnetic structures.

Arguments given by Schwinger [8] in favor of the fact that for the description of SR quantum corrections, under certain conditions, may turn out to be significant also hold true in the case of description of the UR. And here arises the following problem. The point is that a consistent quantum description (in QED) of the radiation processes of charged particles in strong external fields, as a rule, is formulated in the so-called Furry picture [10] and is based on the knowledge of exact solutions of the Dirac or Klein–Gordon equations in such fields. Other known methods, for example, the Schwinger method, are associated with the use of additional approximations. If for the purposes of SR describing exact solutions of the mentioned wave equations with an uniform and constant magnetic field are known and well studied, then for the purposes of describing the UR, solutions in periodic magnetic structures are still unknown. In this regard, we note that in our work [13] we have proposed an approach to describing the quantum properties of the radiation of charged currents that does not require the use of the Furry picture, that is, a complex technique operating with exact solutions. Here electric currents generating the radiation are considered classically, whereas the quantum nature of the radiation is taken into account exactly. Here and in what follows, we call such a way of radiation calculation the semiclassical approximation. Naturally, the semiclassical approximation has its own area of applicability, in particular, it does not take into account the back reaction of the radiation field to charged particles. However, it may be helpful in some cases, for example, it allows one to study one-photon and multi-photon radiations without complicating calculations by using corresponding solutions of the Dirac equation. The effectiveness of the semiclassical approximation was demonstrated by the example of the description of SR.

The article is organized as follows. In Section 2 we describe the semiclassical approximation for treating the radiation from classical currents. In particular, we present exact quantum states of the quantized electromagnetic field interacting with classical currents. These states are used to calculate a probability of many-photon radiation from the vacuum initial state of the electromagnetic field. In this manner, in Section 3, we study characteristics of electromagnetic radiation of a planar undulator in the semiclassical approximation. We find the total radiated energy and its spectral-angular distribution. In Section 4 we compare our results to ones obtained in the framework of classical electrodynamics, discussing differences introduced by accurate accounting for the quantum nature of electromagnetic radiation and present results of some numerical calculations that confirm, in particular, the latter discussion. In Appendix we present the calculation of the radiated energy using an alternative parametrization of the trajectory of electrons moving in a planar undulator.

2. ELECTROMAGNETIC RADIATION OF CLASSICAL CURRENT IN SEMICLASSICAL APPROXIMATION

The semiclassical approximation considered in [13] is based on a possibility to construct exact quantum states of the electromagnetic field interacting with classical currents. With the help of such states one can calculate the probability of the photon emission and derive the spectral-angular distribution of energy emitted in course of the one-photon and multi-photon radiation. Below we present these formulas, the details of the derivation of which the reader can find in the above mentioned [13].

In the general case, the classical four-current $j^\mu(x) = (j^0(x), j^i(x), i = 1, 2, 3)$, interacting with electromagnetic field, affects its quantum states. The differential probability $P(\mathbf{k}_1\lambda_1, ..., \mathbf{k}_N\lambda_N; t)$ of the emission from the vacuum state of N photons each one with the wave vectors $\mathbf{k}_a$ and polarization λ_a, $a = 1, 2, ..., N$, for the time interval t, has the form:

$$P(\mathbf{k}_1\lambda_1,\dots,\mathbf{k}_N\lambda_N;t) = p(\mathbf{k}_1\lambda_1,\dots,\mathbf{k}_N\lambda_N;t)P(0;t),$$
$$P(0;t) = \exp\left(-\sum_{\lambda=1}^{2}\int d\mathbf{k}|y_{\mathbf{k}\lambda}(t)|^2\right), \quad (1)$$
$$p(\mathbf{k}_1\lambda_1,\dots,\mathbf{k}_N\lambda_N;t) = (N!)^{-1}\prod_{a=1}^{N}|y_{\mathbf{k}_a\lambda_a}(t)|^2.$$

Here $P(0; t)$ is the vacuum-to-vacuum transition probability (the probability of a transition without any photon emission), such that the quantity $p(\mathbf{k}_1\lambda_1, ..., \mathbf{k}_N\lambda_N; t)$ can be interpreted as a relative probability of the N photon emission. The functions $y_{\mathbf{k}\lambda}(t)$ are defined as:

$$y_{\mathbf{k}\lambda}(t) = i\sqrt{\frac{4\pi}{\hbar c}}\int_0^t dt'\int j^i(x')f^{i*}_{\mathbf{k}\lambda}(x')d\mathbf{r}',$$
$$f^i_{\mathbf{k}\lambda}(x) = \frac{\exp[-i(k_0ct - \mathbf{kr})]}{\sqrt{2k_0(2\pi)^3}}\epsilon^i_{\mathbf{k}\lambda}, \quad k_0 = |\mathbf{k}|, \quad (2)$$

where $\epsilon^i_{\mathbf{k}\lambda}$ are polarization vectors of photons with the quantum numbers $\mathbf{k}$, λ.[1] They are perpendicular to the wave vector $\mathbf{k}$ and have the properties

$$\epsilon_{\mathbf{k}\lambda}\epsilon^*_{\mathbf{k}\lambda} = \delta_{\lambda\sigma}, \quad \epsilon_{\mathbf{k}\lambda}\mathbf{k} = 0,$$
$$\sum_{\lambda=1}^{2}\epsilon^i_{\mathbf{k}\lambda}\epsilon^{j*}_{\mathbf{k}\lambda} = \delta^{ij} - k^ik^j|\mathbf{k}|^{-2}. \quad (3)$$

[1] Here and in what follows we use the summation convention for dummy indices, i.e., $a^ib^i = \sum_i a^i b^i$, unless explicitly stated otherwise.

In our consideration the introduced vectors are chosen in the following form:

$$\begin{gathered}\mathbf{k} = (k_0 \sin\theta\cos\varphi, k_0 \sin\theta\sin\varphi, k_0\cos\theta),\\ \epsilon_{\mathbf{k}1} = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta),\\ \epsilon_{\mathbf{k}2} = (-\sin\varphi, \cos\varphi, 0),\\ \epsilon_{\mathbf{k}1}\epsilon_{\mathbf{k}1} = \epsilon_{\mathbf{k}2}\epsilon_{\mathbf{k}2} = 1,\\ \epsilon_{\mathbf{k}1}\epsilon_{\mathbf{k}2} = \epsilon_{\mathbf{k}1}\mathbf{k} = \epsilon_{\mathbf{k}2}\mathbf{k} = 0.\end{gathered} \tag{4}$$

The energy of N photons with given quantum numbers $\mathbf{k}_a\lambda_a$ depends only on their momenta $\mathbf{k}_a$, and does not depend on their polarizations; it is equal to

$$W(\mathbf{k}_1\lambda_1,\ldots,\mathbf{k}_N\lambda_N) = \hbar c\sum_{a=1}^{N}|\mathbf{k}_a|. \tag{5}$$

Therefore the energy emitted in course of the process reads:

$$\begin{gathered}W(\mathbf{k}_1\lambda_1,\ldots,\mathbf{k}_N\lambda_N;t)\\ = W(\mathbf{k}_1\lambda_1,\ldots,\mathbf{k}_N\lambda_N)P(\mathbf{k}_1\lambda_1,\ldots,\mathbf{k}_N\lambda_N;t).\end{gathered} \tag{6}$$

To find the energy $W(N;\, t)$, emitted by all N-photon processes, we sum (6) over all possible quantum numbers $\mathbf{k}_a$, λ_a,

$$\begin{gathered}W(N;t) = \hbar c(N!)^{-1}P(0;t)\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\ldots\sum_{\lambda_N=1}^{2}\\ \times\int d\mathbf{k}_1 d\mathbf{k}_2\ldots d\mathbf{k}_N\left[\sum_{b=1}^{N}|\mathbf{k}_b|\right]\prod_{a=1}^{N}|y_{\mathbf{k}_a\lambda_a}(t)|^2.\end{gathered} \tag{7}$$

The RHS of Eq. (7) can be represented as:

$$\begin{gathered}W(N;t) = \frac{A}{(N-1)!}P(0;t)\left(\sum_{\lambda=1}^{2}\int d\mathbf{k}|y_{\mathbf{k}\lambda}(t)|^2\right)^{N-1},\\ A = \hbar c\sum_{\lambda=1}^{2}\int d\mathbf{k}k_0|y_{\mathbf{k}\lambda}(t)|^2.\end{gathered} \tag{8}$$

Summing this quantity over N, we find the total energy $W(t)$ of all emitted photons,

$$\begin{gathered}W(t) = \sum_{N=1}^{\infty}W(N;t) = \hbar c\sum_{\lambda=1}^{2}\int d\mathbf{k}k_0 p_{\mathbf{k}\lambda}(t),\\ p_{\mathbf{k}\lambda}(t) = |y_{\mathbf{k}\lambda}(t)|^2.\end{gathered} \tag{9}$$

3. RADIATION OF PLANAR UNDULATOR IN SEMICLASSICAL APPROXIMATION

Let us consider an electron that moves in a planar undulator along the axis z in the plane xz ($y = 0$), performing transverse oscillations along the axis x with frequency ω_p. The electron dynamics and radiation from the relativistic electron moving in such a device have been first considered in [5] and later in more detail in works [14, 18], see also [19]. The electron current for the case has the form

$$\begin{gathered}j^i(x) = ev^i(t)\delta(x - x(t))\delta(y - y(t))\delta(z - z(t)),\\ v^i(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t)),\\ x(t) = \frac{K}{\gamma}\frac{\lambda_p}{2\pi}\cos(\omega_p t), \quad y(t) = 0,\\ z(t) = c\beta_0 t + \frac{K^2}{\gamma^2}\frac{\lambda_p}{16\pi}\sin(2\omega_p t),\\ \dot{x}(t) = -c\beta_0\frac{K}{\gamma}\sin(\omega_p t), \quad \dot{y}(t) = 0,\\ \dot{z}(t) = c\beta_0\left[1 + \frac{K^2}{4\gamma^2}\cos(2\omega_p t)\right],\\ \omega_p = 2\pi c\beta_0\lambda_p^{-1}, \quad K = \frac{eH\lambda_p}{2\pi m_0 c^2},\\ \beta_0 = \beta\left(1 - \frac{K^2}{4\gamma^2}\right), \quad \beta = v/c,\end{gathered} \tag{10}$$

where the parameter K is the so-called parameter of the undulator strength, λ_p is the length of a single undulator period, γ is the Lorentz factor, v is the particle velocity, β_0 is the average velocity of particle displacement along the axis z, m_0 is the rest mass of the electron, and H is strength of the magnetic field in the undulator. In most cases of interest the parameter K satisfies the inequality $K \le 1$ (weak undulators). For any realistic K, however, the ratio K/γ is very small for a relativistic electron, $K/\gamma \ll 1$, and $\beta_0 \approx \beta$.

Here we calculate the radiation generated by the current (10) using the formulas given in Section 2. Substituting the current $j^i(x)$ into (2), and using representation (4) for wave vector $\mathbf{k}$ and polarization vectors $\epsilon_{\mathbf{k}\lambda}$, we write the functions $y_{\mathbf{k}\lambda}(t)$ for the planar undulator as

$$\begin{gathered}y_{\mathbf{k}\lambda}(t) = iec\beta_0[\hbar c k_0(2\pi)^2]^{-1/2}\\ \times\int_0^t dt' A_\lambda(\theta,\varphi,t')\exp[i\kappa(t')],\\ A_1(\theta,\varphi,t) = -\sin\theta\left[1 + \frac{K^2}{4\gamma^2}\cos(2\omega_p t)\right]\\ -\cos\varphi\cos\theta\frac{K}{\gamma}\sin(\omega_p t),\\ A_2(\theta,\varphi,t) = \sin\varphi\frac{K}{\gamma}\sin(\omega_p t),\\ \kappa(t) = ck_0 r(1 - \beta_0\cos\theta) - u\cos\omega_p t - s\sin(2\omega_p t),\\ u = \frac{K}{\gamma}\frac{\lambda_p}{2\pi}k_0\sin\theta\cos\varphi, \quad s = k_0\frac{K^2}{\gamma^2}\frac{\lambda_p}{16\pi}\cos\theta,\end{gathered} \tag{11}$$

and the functions $p_{\mathbf{k}\lambda}(t)$ as

$$p_{\mathbf{k}\lambda}(t) = \frac{e^2c^2\beta_0^2}{\hbar c k_0(2\pi)^2}\left|\int_0^t dt' A_\lambda(\theta,\varphi,t')\exp[i\kappa(t')]\right|^2. \quad (12)$$

Next, we transform the exponent in Eq. (12) using the following expansions of trigonometric functions in terms of the Bessel functions [20],

$$\exp(-iu\cos\omega_p t) = \sum_{n=-\infty}^{+\infty}(-i)^n J_n(u)\exp(-in\omega_p t),$$
$$\exp(-is\sin 2\omega_p t) = \sum_{n=-\infty}^{+\infty} J_m(s)\exp(-i2m\omega_p t), \quad (13)$$

to obtain

$$p_{\mathbf{k}\lambda}(t) = \frac{e^2c^2\beta_0^2}{\hbar c k_0(2\pi)^2}\left|\sum_{n,m=-\infty}^{\infty} J_n(u)J_m(s)e^{-in\pi/2}\right.$$
$$\left.\times\int_0^t dt' A_\lambda(\theta,\varphi,t')\exp[i\omega_p t' R_{nm}]\right|^2, \quad (14)$$
$$R_{nm} = ck_0\omega_p^{-1}(1-\beta_0\cos\theta) - n - 2m.$$

The integration over dt' can be carried out explicitly. To this end we define the functions $B_{1,2,3}$ (R_{nm}, t),

$$B_1(R_{nm},t) = \int_0^t dt'\exp[i\omega_p t' R_{nm}]$$
$$= t\exp(iR_{nm}\omega_p t/2)\mathrm{sinc}(R_{nm}\omega_p t/2),$$
$$\mathrm{sinc}(x) = \sin x/x,$$

$$B_2(R_{nm},t) = \int_0^t dt'\sin(\omega_p t')\exp[i\omega_p t' R_{nm}]$$
$$= -2^{-1}it\{\exp[i(R_{nm}+1)\omega_p t/2]\mathrm{sinc}[(R_{nm}+1)\omega_p t/2] \quad (15)$$
$$-\exp[i(R_{nm}-1)\omega_p t/2]\mathrm{sinc}[(R_{nm}-1)\omega_p t/2]\},$$

$$B_3(R_{nm},t) = \int_0^t dt'\cos(2\omega_p t')\exp[i\omega_p t' R_{nm}]$$
$$= 2^{-1}t\{\exp[i(R_{nm}+2)\omega_p t/2]\mathrm{sinc}[(R_{nm}+2)\omega_p t/2]$$
$$+\exp[i(R_{nm}-2)\omega_p t/2]\mathrm{sinc}[(R_{nm}-2)\omega_p t/2]\}.$$

Substituting (14) in (9) and taking into account (15), we obtain the expression for the total energy $W(t)$,

$$W(t) = \left(\frac{ec\beta_0}{2\pi}\right)^2\int_0^\infty k_0^2 dk_0\int_0^\pi \sin\theta d\theta$$
$$\times\int_0^{2\pi} d\varphi\left(\left|\sum_{n,m=-\infty}^{\infty} J_n(u)J_m(s)e^{-in\pi/2}\right.\right.$$
$$\times\left\{\left[\frac{K^2}{4\gamma^2}B_1(R_{nm},t) + B_3(R_{nm},t)\right]\sin\theta\right. \quad (16)$$
$$\left.\left.+B_2(R_{nm},t)\frac{K}{\gamma}\cos\varphi\cos\theta\right\}\right|^2$$
$$\left.+\left|\sum_{n,m=-\infty}^{\infty} J_n(u)J_m(s)e^{-in\pi/2}B_2(R_{nm},t)\frac{K}{\gamma}\sin\varphi\right|^2\right).$$

4. COMPARISON WITH THE CLASSICAL TREATMENT

Comparing expression (16) obtained in the semiclassical approximation with the spectral-angular distribution of the radiated energy obtained in the framework of classical electrodynamics (see for example [19]), one can see that in the semiclassical approximation the angular distribution for the photons with the polarization $\lambda = 1$ and $\lambda = 2$ is different from the one given by the classical treatment, while spectral distributions (note that the quantity ck_0 corresponds to photon's frequency) are the same. We note that in the main direction of the radiation $\theta = 0$, $\varphi = 0$ (on axis radiation) the semiclassical approximation and the classical treatment give the same results.

For sufficiently large time period t, i.e., for an undulator with a large number of sections, the radiation spectrum is determined by the integrand of Eq. (16) and degenerates into a set of narrow peaks at the values of k_0 that are defined by the conditions:

$$R_{nm} = 0, \quad R_{nm}+1 = 0, \quad R_{nm}-1 = 0,$$
$$R_{nm}+2 = 0, \quad R_{nm}-2 = 0. \quad (17)$$

In particular, the case $t \to \infty$ formally corresponds to an infinite undulator. In this case one can use the relation:

$$\lim_{t\to\infty}\frac{\sin xt}{x} \to \pi\delta(x). \quad (18)$$

This means that for the infinite undulator the energy spectrum is concentrated in points that are defined by conditions (17).

Let us compare the spectral-angular distribution obtained in the semiclassical approximation with the one obtained in the framework of the classical electrodynamics. It follows from Eq. (16) that

$$\frac{d^2W(t)}{d(ck_0)d\Omega} = c^{-1}\left(\frac{eck_0\beta_0}{2\pi}\right)^2 \times\left\{\left|\sin\theta\left[C_1(t)+\frac{K^2}{4\gamma^2}C_3(t)\right] + \cos\varphi\cos\theta\frac{K}{\gamma}C_2(t)\right|^2 + \left|\sin\varphi\frac{K}{\gamma}C_2(t)\right|^2\right\}, \quad d\Omega = \sin\theta d\theta d\varphi, \tag{19}$$

where the functions $C_{1,2,3}(t)$ are:

$$C_1(t) = \int_0^t dt' e^{i\kappa(t')},$$
$$C_2(t) = \int_0^t dt' \sin(\omega_p t')e^{i\kappa(t')}, \tag{20}$$
$$C_3(t) = \int_0^t dt' \cos(2\omega_p t')e^{i\kappa(t')}.$$

At the same time, the spectral-angular distribution of the radiated energy obtained in the framework of the classical electrodynamics has the form:

$$\frac{d^2W(t)}{d(ck_0)d\Omega} = c^{-1}\left(\frac{eck_0\beta_0}{2\pi}\right)^2 |D_1\mathbf{x} + D_2\mathbf{y} + D_3\mathbf{z}|^2,$$
$$D_1 = \frac{K}{\gamma}\tilde{C}_2 - \frac{K}{\gamma}\tilde{C}_4\cos^2\varphi\sin^2\theta + \cos\varphi\sin\theta\cos\theta\left[\tilde{C}_1 + \frac{K^2}{4\gamma}\tilde{C}_3\right],$$
$$D_2 = -\frac{K}{\gamma}\tilde{C}_2\sin\varphi\cos\varphi\sin^2\theta + \cos\varphi\sin\theta\cos\theta\left[\tilde{C}_1 + \frac{K^2}{4\gamma}\tilde{C}_3\right], \tag{21}$$
$$D_3 = -\frac{K}{\gamma}\tilde{C}_4\cos\varphi\sin\theta\cos\theta + (\cos^2\theta - 1)\left[\tilde{C}_1 + \frac{K^2}{4\gamma}\tilde{C}_3\right],$$

where **x**, **y**, and **z** are unit vectors in the direction of corresponding coordinate axes, and the functions $\tilde{C}_{1,2,3,4}$ are:

$$\tilde{C}_1 = \int_{-t/2}^{t/2} dt' e^{-i\kappa(t')}, \quad \tilde{C}_2 = \int_{-t/2}^{t/2} dt' \sin(\omega_p t')e^{-i\kappa(t')},$$
$$\tilde{C}_3 = \int_{-t/2}^{t/2} dt' \cos(2\omega_p t')e^{-i\kappa(t')}, \tag{22}$$
$$\tilde{C}_4 = \int_{-t/2}^{t/2} dt' \cos(\omega_p t')e^{-i\kappa(t')}.$$

We see that distributions (19) and (21) obtained by the semiclassical and classical methods, respectively, are essentially different in the general case. Nevertheless, both expressions (19) and (21) can be reduced to the same form

$$\frac{d^2W(t)}{d(ck_0)d\Omega} \approx c^{-1}\left(\frac{e\beta_0 ck_0}{2\pi}\right)^2 \times\left[\theta^2\cos^2\varphi|C_1|^2 + \theta^2\sin^2\varphi|C_1|^2 + \frac{K^2}{\gamma^2}|C_2|^2 + \frac{K}{\gamma}\theta\cos\varphi(C_1C_2^* + C_1^*C_2)\right] \tag{23}$$

for planar undulators, for which $K/\gamma \ll 1$ and for $\theta \ll 1$, taking at the same time into account that the electron flight time through the undulator is $t = 2\pi N\omega_p^{-1}$. The numerical analysis of both formulas also confirms the latest findings. The corresponding results are represented in Fig. 1.

We note that often, while describing polarization properties of the radiated energy $W(t)$ one uses the so-called σ and π radiation modes. We recall that the σ-mode characterizes the radiation emitted perpendicular to the magnetic field direction while the π-mode characterizes the radiation emitted parallel to the magnetic field direction. For a planar undulator the σ-mode is formed by components of the electric field in the x and z directions, while the π-mode is formed by electric field component in the y (see, e.g., [19] for details).

Let us determine the contributions corresponding to the σ and π modes in Eq. (16). This can be done as follows. First, let us note that the scalar product of the electric current $j^i(x)$ with polarization vectors $\epsilon^i_{\mathbf{k}\lambda}$ can be interpreted as the projection of the electric current vector on the directions of polarization vectors $\epsilon^i_{\mathbf{k}\lambda}$,

$$\mathbf{j}_{\mathbf{k}\lambda}(x) = [j^j(x)\epsilon^j_{\mathbf{k}\lambda}]\epsilon_{\mathbf{k}\lambda}, \quad |\mathbf{j}_{\mathbf{k}\lambda}(x)| = [j^j(x)\epsilon^j_{\mathbf{k}\lambda}]. \tag{24}$$

The vectors $j^i_{\mathbf{k}\lambda}(x) = \mathbf{j}_{\mathbf{k}1}(x)$ can be decomposed into components in the directions of the axes x, y, and z; we also recall that the vector $\epsilon_{\mathbf{k}2}$ is situated on the xy plane,

$$\mathbf{j}_{\mathbf{k}1}(x) = [j^j(x)\epsilon^j_{\mathbf{k}\lambda}](\mathbf{x}\cos\varphi\cos\theta + \mathbf{y}\sin\varphi\cos\theta - \mathbf{z}\sin\theta),$$
$$\mathbf{j}_{\mathbf{k}2}(x) = [j^j(x)\epsilon^j_{\mathbf{k}\lambda}](-\mathbf{x}\sin\varphi + \mathbf{y}\cos\varphi). \tag{25}$$

It can be seen from (2) that the functions $|y_{\mathbf{k}\lambda}(t)|^2$ can be represented as

$$|y_{\mathbf{k}1}(t)|^2 = |y_{\mathbf{k}1}(t)|^2 \times(\mathbf{x}^2\cos^2\varphi\cos^2\theta + \mathbf{y}^2\sin^2\varphi\cos^2\theta + \mathbf{z}^2\sin^2\theta), \tag{26}$$
$$|y_{\mathbf{k}2}(t)|^2 = |y_{\mathbf{k}2}(t)|^2(\mathbf{x}^2\sin^2\varphi + \mathbf{y}^2\cos^2\varphi),$$

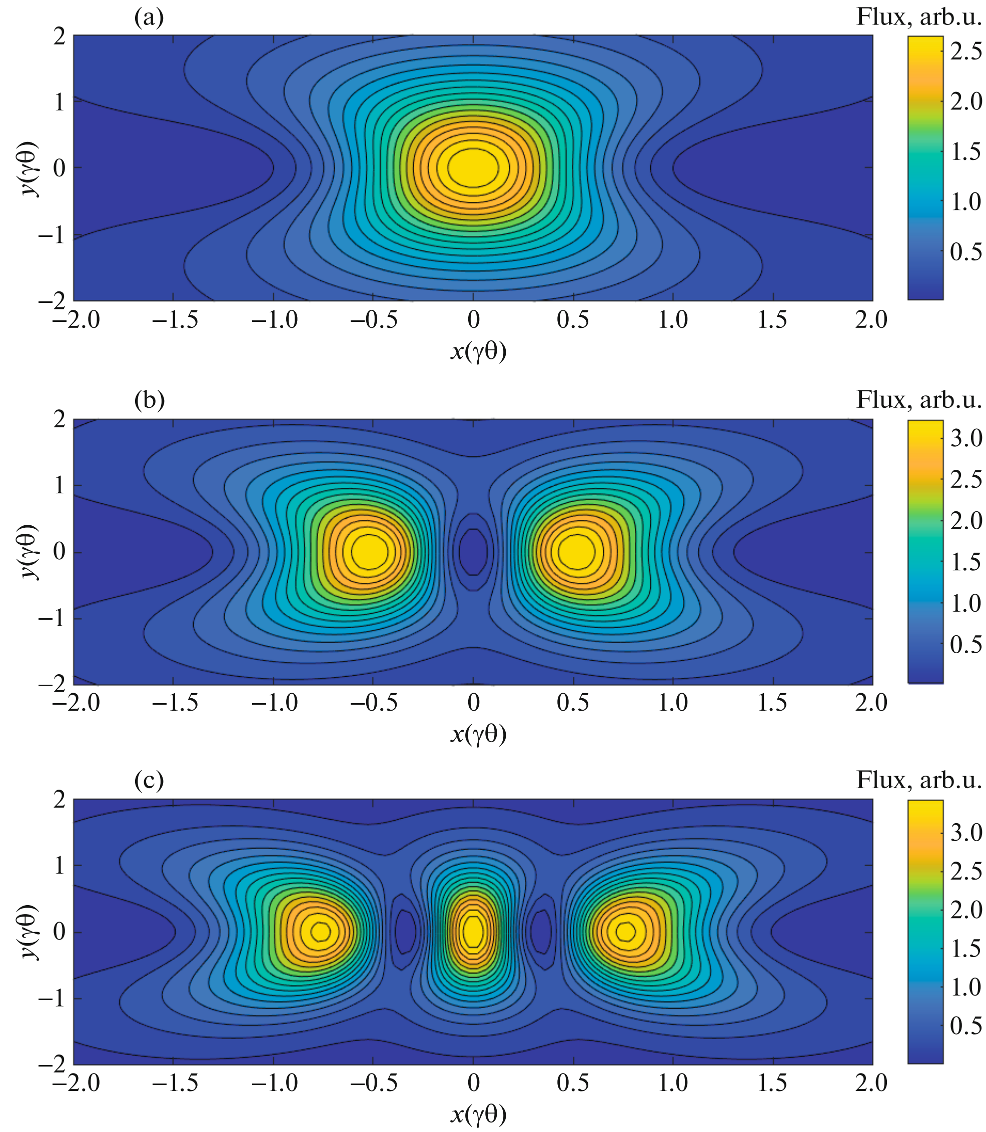

Fig. 1. The total spatial angular distribution of total radiated energy for the first three harmonic calculated in semiclassical approximation.

where we kept squares of unit vectors **x**, **y**, **z** to keep track of the polarization directions. Since the magnetic field in a planar undulator is directed along the axis y, contributions from the terms multiplied by $\mathbf{x}^2$ and $\mathbf{z}^2$ correspond to σ-mode, while contributions from the terms multiplied by $\mathbf{y}^2$ correspond to π-mode.

Thus, the expressions for the energy radiated in σ- and π-mode have the form

$$W_\sigma(t) = \left(\frac{ec\beta_0}{2\pi}\right)^2 \int_0^\infty k_0^2 dk_0 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi p_{\mathbf{k}\sigma}(t),$$
$$W_\pi(t) = \left(\frac{ec\beta_0}{2\pi}\right)^2 \int_0^\infty k_0^2 dk_0 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi p_{\mathbf{k}\pi}(t), \quad (27)$$
$$p_{\mathbf{k}\sigma}(t) = |y_{\mathbf{k}1}(t)|^2(\cos^2\varphi\cos^2\theta + \sin^2\theta) + |y_{\mathbf{k}2}(t)|^2\sin^2\varphi,$$
$$p_{\mathbf{k}\pi}(t) = |y_{\mathbf{k}1}(t)|^2\sin^2\varphi\cos^2\theta + |y_{\mathbf{k}2}(t)|^2\cos^2\varphi.$$

It is easy to verify that the following relations hold true,

$$p_{\mathbf{k}1}(t) + p_{\mathbf{k}2}(t) = p_{\mathbf{k}\sigma}(t) + p_{\mathbf{k}\pi}(t),$$
$$W(t) = W_\sigma(t) + W_\pi(t). \quad (28)$$

We demonstrate our numerical analysis of the energy distribution for the first three harmonics in Figs. 2, 3, 4. Note that while the total radiated energy distribution coincides in main with the classical one, the semiclassical approach shows different and more detailed distribution of the radiated energy in π and σ modes.

In conclusion, note that the expressions for the radiation characteristics obtained in the semiclassical approximation, even in those cases when they do not differ quantitatively from the corresponding classical expressions, have a much simpler analytical form and their derivation is technically much simpler.

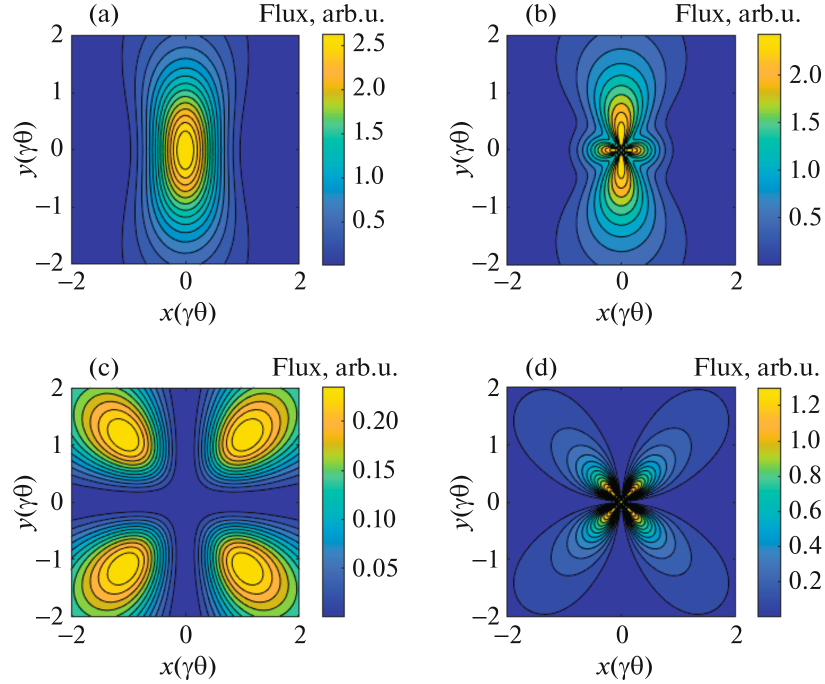

Fig. 2. Spatial angular distribution of σ (top) and π (bottom) modes for the first harmonic calculated by classical method (left) and in semiclassical approximation (right).

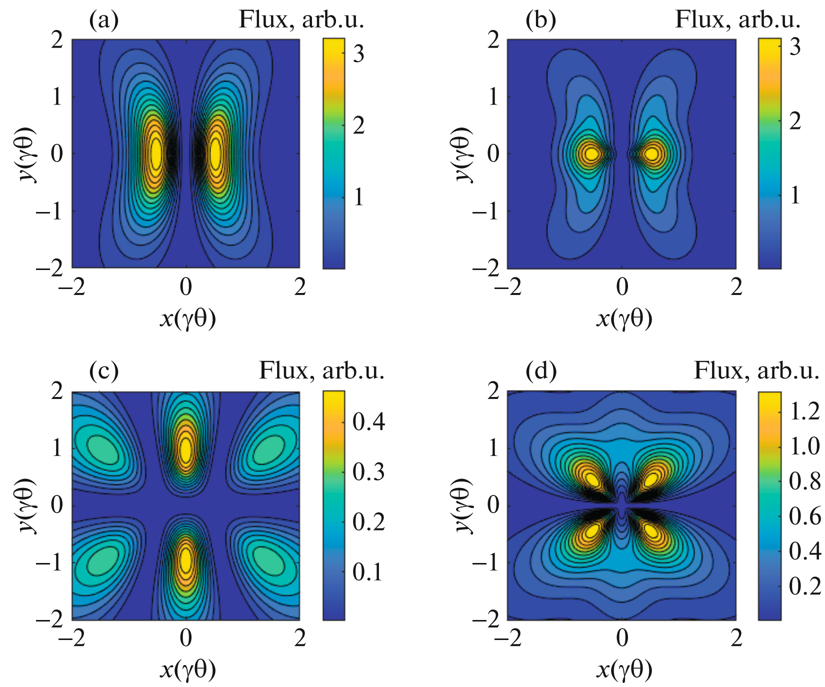

Fig. 3. Spatial angular distribution of σ (top) and π (bottom) modes for the second harmonic calculated by classical method (left) and in semiclassical approximation (right).

APPENDIX

Calculating Radiation of Planar Undulator in Semiclassical Approximation Using an Alternative Parametrization of Trajectory

In [15] a special trajectory for electrons moving in a plane undulator were used to calculate their radiation. It is supposed that the trajectory of electrons is plane and symmetrical relatively to the axis x, and consists of circular arcs of length l and radius R. This representation provides an alternative parametrization of the electron trajectory in plane undulator. In this

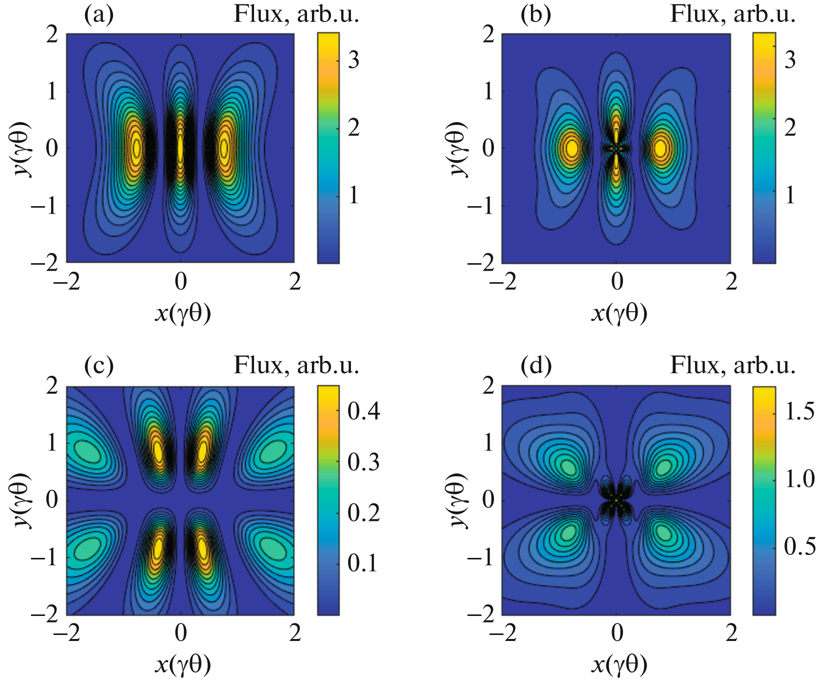

Fig. 4. Spatial angular distribution of σ (top) and π (bottom) modes for the third harmonic calculated by classical method (left) and in semiclassical approximation (right).

appendix we apply the semiclassical method to calculate the radiation from electrons moving in such a trajectory.

Consider electrons moving in a periodic magnetic field parallel to the axis z, such that in each period the magnetic field is homogeneous and constant. The undulator is assumed to be of infinite length. A length l of each individual arc is related to the effective radius of curvature R via the so-called injection angle $\alpha = l/R$, $0 < \alpha < \pi$. The velocity of the electrons is $v = c\beta = \omega R$, where ω is the angular velocity. The electrons move along the axis x with an average velocity $v_0 = c\bar{\beta}$, and perform periodic oscillations along the axes x and y. This implies

$$\bar{\beta} = \beta \operatorname{sinc}(\alpha/2), \quad T = 2\pi\omega_0^{-1} = 2\alpha\omega^{-1}, \\ \omega_0 = \pi\omega\alpha^{-1} = \pi\beta c l^{-1}. \tag{29}$$

The trajectory at the time interval $(0, T)$ can be presented as [15]

$$x(t) = \begin{cases} R[\sin(\alpha/2) + \sin(\omega t - \alpha/2)], & t \in T_1 \\ R[3\sin(\alpha/2) + \sin(\omega t - \alpha/2)], & t \in T_2, \end{cases} \\ y(t) = \begin{cases} R[\cos(\omega t - \alpha/2) - \cos(\alpha/2)], & t \in T_1 \\ R[\cos(\alpha/2) - \cos(\omega t - \alpha/2)], & t \in T_2, \end{cases} \tag{30}$$

where the time intervals T_1 and T_2 are defined as

$$T_1 = [0, T/2]; \quad T_2 = (T/2, T]. \tag{31}$$

The current $j^i(x)$ formed by electrons moving in the trajectory (30) has the form

$$j^i(x) = ev^i(t)\delta(x - x(t))\delta(y - y(t))\delta(z - z(t)), \\ v^i(t) = \dot{r}^i(t) = (\dot{x}(t), \dot{y}(t), 0), \\ \dot{x}(t) = \begin{cases} \omega R\cos[\alpha/2 - \omega t], & t \in T_1 \\ \omega R\cos[3\alpha/2 - \omega t], & t \in T_2, \end{cases} \\ \dot{y}(t) = \begin{cases} \omega R\sin[\alpha/2 - \omega t], & t \in T_1 \\ -\omega R\sin[3\alpha/2 - \omega t], & t \in T_2. \end{cases} \tag{32}$$

Let us calculate the radiation energy during a single period T. The functions $y_{\mathbf{k}\lambda}(t)$ at the interval $t = T$ can be calculated as

$$y_{\mathbf{k}\lambda}(T) = y_{\mathbf{k}\lambda}(T_1) + y_{\mathbf{k}\lambda}(T_2). \tag{33}$$

Using the definitions (4) for the wave vector $\mathbf{k}$ and polarization vectors $\boldsymbol{\epsilon}_{\mathbf{k}\lambda}$, we obtain

$$y_{\mathbf{k}1}(T_j) = z_j e^{i\phi_j}\cos\theta \int_{\tau_j^{\text{in}}}^{\tau_j^{\text{out}}} \omega^{-1} d\tau_j \\ \times [c\bar{\beta}\cos\varphi + \omega R\cos\tau_j]\exp[i\varkappa(\tau_j)], \\ y_{\mathbf{k}2}(T_j) = -z_j e^{i\phi_j} \int_{\tau_j^{\text{in}}}^{\tau_j^{\text{out}}} \omega^{-1} d\tau_j \\ \times [c\bar{\beta}\sin\varphi + (-1)^{j-1}\omega R\sin\tau_j]\exp[i\varkappa(\tau_j)], \tag{34}$$

where we used the notation

$$\varkappa(\tau_j) = ck_0\omega^{-1}\tau_j - \xi\sin\tau_j, \quad \xi = Rk_0\sin\theta,$$
$$z_j = ie\exp[-i\xi f_j(\varphi,\alpha)][\hbar ck_0(2\pi)^2]^{-1/2}, \quad j = 1,2,$$
$$f_1(\varphi,\alpha) = \sin(\alpha/2)\cos\varphi - \cos(\alpha/2)\sin\varphi,$$

$$f_2(\varphi,\alpha) = \cos(\alpha/2)\sin\varphi + 3\sin(\alpha/2)\cos\varphi,$$
$$\tau_1 = \omega t - \alpha/2 + \varphi, \quad \tau_2 = \omega t - 3\alpha/2 - \varphi, \tag{35}$$

$$\tau_1^{\text{in}} = \varphi - \alpha/2, \quad \tau_1^{\text{out}} = \varphi + \alpha/2,$$
$$\tau_2^{\text{in}} = -\varphi - \alpha/2, \quad \tau_2^{\text{out}} = \alpha/2 - \varphi,$$
$$\phi_1 = \omega^{-1}\kappa(\alpha/2 - \varphi), \quad \phi_2 = \omega^{-1}\kappa(\varphi + 3\alpha/2).$$

Using the known expansions of trigonometric functions in terms of Bessel functions,

$$\exp(-i\xi\sin\tau) = \sum_{n=-\infty}^{+\infty} J_n(\xi)\exp(-in\tau),$$

$$\sin\tau\exp(-i\xi\sin\tau) = i\sum_{n=-\infty}^{+\infty} J_n'(\xi)\exp(-in\tau), \tag{36}$$

$$\cos\tau\exp(-i\xi\sin\tau) = \sum_{n=-\infty}^{+\infty} \frac{n}{\xi}J_n(\xi)\exp(-in\tau),$$

we rewrite (34) as

$$y_{\mathbf{k}1}(T_j) = z_j e^{i\phi_j}\sum_{n=-\infty}^{+\infty}\cos\theta[n(k_0\sin\theta)^{-1} + \omega^{-1}c\overline{\beta}\cos\varphi]J_n(\xi)K_n(T_j),$$
$$y_{\mathbf{k}2}(T_j) = -z_j e^{i\phi_j}\sum_{n=-\infty}^{+\infty}[\omega^{-1}c\overline{\beta}\sin\varphi J_n(\xi) + (-1)^{j-1}iRJ_n'(\xi)]K_n(T_j), \tag{37}$$

where functions $K_n(T_j)$ have the form

$$K_n(T_j) = \int_{\tau_j^{\text{in}}}^{\tau_j^{\text{out}}} d\tau_j\exp[i(\omega^{-1}ck_0 - n)\tau_j]$$
$$= \exp[(-1)^j i(\omega^{-1}ck_0 - n)\varphi]\frac{\sin[\alpha(\omega^{-1}ck_0 - n)/2]}{(\omega^{-1}ck_0 - n)/2}. \tag{38}$$

The functions $p_{\mathbf{k}\lambda}(T)$ on the time interval T have the form

$$p_{\mathbf{k}\lambda}(T) = |y_{\mathbf{k}\lambda}(T)|^2 = |y_{\mathbf{k}\lambda}(T_1) + y_{\mathbf{k}\lambda}(T_2)|^2. \tag{39}$$

Substituting expressions (37) into (39), we obtain

$$p_{\mathbf{k}1}(T) = \left|\sum_{n=-\infty}^{+\infty}[n(k_0\sin\theta)^{-1} + \omega^{-1}c\overline{\beta}\cos\varphi]\cos\theta J_n(\xi)S_1\right|^2,$$
$$p_{\mathbf{k}2}(T) = \left|\sum_{n=-\infty}^{+\infty}[\omega^{-1}c\overline{\beta}\sin\varphi J_n(\xi)S_1 + iRJ_n'(\xi)S_2]\right|^2, \tag{40}$$
$$S_1 = z_1e^{i\phi_1}K_n(T_1) + z_2e^{i\phi_2}K_n(T_2),$$
$$S_2 = z_1e^{i\phi_1}K_n(T_1) - z_2e^{i\phi_2}K_n(T_2).$$

The total radiated energy $W(T)$ takes the form

$$W(T) = \int_0^\infty k_0^2dk_0\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi[p_{\mathbf{k}1}(T) + p_{\mathbf{k}2}(T)], \tag{41}$$

where $p_{\mathbf{k}1}(T)$ and $p_{\mathbf{k}2}(T)$ are given by Eq. (40).

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