Reducible Stueckelberg symmetry and dualities

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A B S T R A C T

We propose a general procedure for iterative inclusion of Stueckelberg fields to convert the theory into gauge system being equivalent to the original one. In so doing, we admit reducibility of the Stueckelberg gauge symmetry. In this case, no pairing exists between Stueckelberg fields and gauge parameters, unlike the irreducible Stueckelberg symmetry. The general procedure is exemplified by the case of Proca model, with the third order involutive closure chosen as the starting point. In this case, the set of Stueckelberg fields includes, besides the scalar, also the second rank antisymmetric tensor. The reducible Stueckelberg gauge symmetry is shown to admit different gauge fixing conditions. One of the gauges reproduces the original Proca theory, while another one excludes the original vector and the Stueckelberg scalar. In this gauge, the irreducible massive spin one is represented by antisymmetric second rank tensor obeying the third order field equations. Similar dual formulations are expected to exist for the fields of various spins.

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1. Introduction

Since the original Stueckelberg's work [1], the idea remains attractive for decades concerning inclusion of auxiliary fields into the action in such a way that modified theory becomes gauge invariant while it is still equivalent to the original one. The reviews and further references can be found in [2], [3].

In the constrained Hamiltonian formalism, the Stueckelberg idea has transformed into a method of converting the second class constraints into the first class ones [4], [5]. The conversion is achieved by extending the phase space by extra dimensions, that can be understood as introduction of Stueckelberg fields. The local existence theorem for the conversion procedure has been proven in the article [6], the global proof of the conversion existence can be found in [7]. The starting point of the Hamiltonian conversion is a complete system of the constraints including primary and secondary ones of all the generations. The conversion variable is assigned to every second class constraint. Given the complete system of constraints, the Hamiltonian conversion works as a systematic iterative procedure which is proven unobstructed. Unlike the Hamiltonian counterpart, the common practice of including Stueckelberg fields in Lagrangian formalism seems more art than science. Most often this works as a "Stueckelberg trick", which implies that the action is split into gauge and non-gauge parts. The Stueckelberg gauge symmetry of the original fields is

assumed to remain the same as for the gauge invariant part of the action, while the transformations of Stueckelberg fields are chosen to compensate the non-invariance of the rest part. The choice of this split into gauge and non-gauge parts is an art, and it can be ambiguous. It is even unclear, why such a split is always possible. From the Hamiltonian perspective, this would mean to assume each second class constraint to be decomposed into the first class part and the "symmetry breaking part". The Hamiltonian conversion method proceeds from any complete set of constraints, not assuming the possibility of any decomposition of the constraints.

Recently, a systematic procedure has been proposed for covariant inclusion of the Stueckelberg fields [8] in Lagrangian formalism. The starting point for the method is the *involutive closure* of the original Lagrangian system of field equations. The original equations can be non-involutive, i.e. they can admit the lower order consequences. Completion of the system of the field equations by their consequences is understood as an involutive closure, if the completed system does not admit any further lower order consequences. In principle, the involutive closure can include also the higher order consequences. Completion of the Hamiltonian constrained system by the secondary constraints is an example of the involutive closure. The involutively closed form of the field equations allows one to count the degree of freedom number in an explicitly covariant manner [9]. The procedure of the article [8] allows one to iteratively include Stueckelberg fields for any field theory proceeding from the involutive closure of the original Lagrangian equations, and it is proven to be unobstructed. This procedure implies inclusion of independent consequences into

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the involutive closure of Lagrangian equations. Given this starting point, one arrives at the irreducible Stueckelberg gauge symmetry.

In this article, we consider inclusion of Stueckelberg fields proceeding from the involutive closure which involves a reducible set of consequences of Lagrangian equations. This leads to two main distinctions from the case of independent consequences. First, there is no pairing anymore between the Stueckelberg fields and gauge parameters. Second, the Stueckelberg gauge symmetry turns out reducible. There are no obstructions to inclusion of the Stueckelberg fields in the reducible case, much like to the irreducible one. To exemplify the general procedure, we consider the third order involutive closure of the Proca equations when the original equations are also complemented, besides the first order consequence, by the antisymmetric combinations of the derivatives of the Lagrangian equations. This leads to inclusion, besides the usual Stueckelberg scalar, of the Stueckelberg field, being the second rank antisymmetric tensor. Full Stueckelberg symmetry mixes the original vector with all the Stueckelberg fields. This reducible gauge symmetry admits different gauge fixing conditions. The simplest gauge kills all the Stueckelberg fields reducing the dynamics to the original Proca equations. The alternative gauge fixing condition is also admissible such that kills the Stueckelberg scalar *and the original vector field*, while all the dynamics is described by the antisymmetric tensor $B^{\mu\nu}$ obeying the third order equation,

$$(\Box + m^2)\partial_\nu B^{\mu\nu} = 0, \tag{1}$$

with appropriate gauge fixing for gauge symmetry[1] of $B^{\mu\nu}$. By itself, this non-Lagrangian equation, being the gauge fixed form of the reducible Stueckelberg Lagrangian system, describes the irreducible massive spin 1, much like the original Proca equation. The reason is obvious: the Proca model is equivalent to the Klein-Gordon equation supplemented by the transversality condition,

$$(\Box + m^2)A^\mu = 0, \qquad \partial_\mu A^\mu = 0. \tag{2}$$

In Minkowski space, any transverse vector is a divergence of the antisymmetric tensor,

$$\partial_\mu A^\mu = 0 \quad \Leftrightarrow \quad \exists B^{\mu\nu} = -B^{\nu\mu} : A^\mu = \partial_\nu B^{\mu\nu}. \tag{3}$$

In a sense, $B^{\mu\nu}$ is a "potential" for the transverse vector A^μ. The non-Lagrangian equations (1) can be viewed as a reformulation of the Proca model in terms of the potential, such that automatically accounts for the transversality condition. Under the proposed procedure of inclusion of the Stueckelberg fields, both dual formulations, (1) and (2), are included into a uniform Lagrangian theory even though one of them is non-Lagrangian by itself. Imposing appropriate gauge fixing conditions, one can switch from the vector formulation to the dual one, and vice versa. As explained in the Conclusion, it seems to be a general phenomenon which extends to other representations and goes beyond the free level.

The article is organized as follows. In the next section, the general scheme of inclusion of the Stueckelberg fields is outlined for the case of reducible Stueckelberg gauge symmetry. In Section 3, the general procedure is exemplified by unconventional inclusion of Stueckelberg fields in the Proca model such that leads to reducible gauge symmetry. The results and further perspectives are discussed in the Conclusion.

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[1] For the details of gauge symmetry and gauge fixing of the Stueckelberg field, see in the Section 3.

2. Inclusion of Stueckelberg fields with reducible gauge symmetry

As a preliminary, let us explain the strategy of including Stueckelberg fields implemented in this section. First, the Lagrangian equations are complemented by the consequences such that the entire system is involutive. Once the completed system is non-Lagrangian, the second Noether theorem does not apply, and the gauge identities arise, being unrelated to the gauge symmetry. The general structure of gauge algebra is known for not necessarily Lagrangian field equations [10], [11]. For the case when the non-Lagrangian system is a completion of the Lagrangian one, the gauge algebra has some specifics which are detailed as the second step. As the third step, we introduce the Stueckelberg fields with two goals. First, the involutive system should be zero order of the expansion of Lagrangian Stueckelberg equations. Second, the gauge identities of the involutive closure of the original system should be reproduced as zero order (in Stueckelberg fields) of Noether identities for Stueckelberg action. This defines zero order of gauge symmetry generators and the first order of the action. The existence of all the higher orders can be proven along the similar lines to the irreducible case [8].

In this section, we use the condensed notation. All the condensed indices are supposed to include numerical labels and the space-time points. Summation over the condensed index implies integration over *x*. The partial derivatives are understood as variational.

Consider a theory of fields ϕ^i with the action $S(\phi)$. Lagrangian equations read

$$\partial_i S(\phi) = 0. \tag{4}$$

In this article, we consider a theory where the original action does not have gauge symmetry. This means that any identity between the field equations (4) has a trivial generator which vanishes on shell

$$\kappa^i \partial_i S \equiv 0 \quad \Leftrightarrow \quad \exists E^{ij} = -E^{ji} : \quad \kappa^i = E^{ij}\partial_j S. \tag{5}$$

Inclusion of the Stueckelberg fields in the gauge invariant actions will be considered elsewhere.

Let us complement the field equations (4) by their differential consequences,

$$\tau_\alpha(\phi) = -\Gamma_\alpha{}^i(\phi)\partial_i S(\phi), \tag{6}$$

where $\Gamma_\alpha{}^i(\phi)$ are supposed to be local differential operators. The generators Γ of the consequences are considered equivalent if they lead to the same τ. Hence, the equivalence relation reads

$$\Gamma_\alpha{}^i \sim \Gamma'_\alpha{}^i \Leftrightarrow \Gamma_\alpha{}^i - \Gamma'_\alpha{}^i = E_\alpha{}^{ij}\partial_j S, \ E_\alpha{}^{ij} = -E_\alpha{}^{ji}. \tag{7}$$

The completed system

$$\partial_i S(\phi) = 0, \quad \tau_\alpha(\phi) = 0 \tag{8}$$

is assumed involutively closed, i.e. all the lower order consequences are already contained among equations (8). Obviously, the involutive closure (8) is equivalent to the original system, because all their solutions coincide. By construction, the involutively closed system enjoys gauge identities

$$\Gamma_\alpha{}^i(\phi)\partial_i S(\phi) + \tau_\alpha(\phi) \equiv 0, \tag{9}$$

while there are no gauge symmetry. Let us assume the set of the generators $\Gamma_\alpha{}^i$ of consequences (6) is over-complete,

$$Z_A{}^\alpha \Gamma_\alpha{}^i = E_A{}^{ij}\partial_j S, \quad E_A{}^{ij} = -E_A{}^{ji}, \tag{10}$$

i.e. certain combinations of Γ's reduce to the trivial gauge generators (5). This results in the identities between the consequences (6):

$$Z_A{}^\alpha \tau_\alpha \equiv 0 . \tag{11}$$

The generators of identities are considered equivalent if they differ by the trivial generator vanishing on shell,

$$Z_A{}^\alpha \sim Z'_A{}^\alpha \Leftrightarrow Z_A{}^\alpha - Z'_A{}^\alpha = E_A{}^{\alpha\beta}\tau_\beta ,$$
$$E_A{}^{\alpha\beta} = -E_A{}^{\beta\alpha} . \tag{12}$$

The operators $Z_A{}^\alpha$ are assumed to constitute the generating set for the null-vectors of the consequences τ_α, i.e. $Z^\alpha \tau_\alpha \equiv 0 \Leftrightarrow Z^\alpha = \zeta^A Z_A{}^\alpha$. The identities (11) can admit further reducibility,

$$\exists Z_{1A_1}{}^A : Z_{1A_1}{}^A Z_A{}^\alpha = E_{A_1}{}^{\alpha\beta}\tau_\beta ,$$
$$E_{A_1}{}^{\alpha\beta} = -E_{A_1}{}^{\beta\alpha} , \tag{13}$$

i.e. certain combinations of the identity generators $Z_A{}^\alpha$ reduce to the trivial null-vectors (12). In principle, the generating set of the second level null-vectors $Z_{1A_1}{}^A$ can be over-complete in its own turn. In this article, we do not consider this option assuming no further reducibility.

The set of identities (9), (11) between the equations of involutive closure (8) is assumed complete. This means, any set of identities, labeled by some condensed index I, reduces to the linear combination of identities (9), (11),

$$\Lambda_I{}^i \partial_i S + \Lambda_I{}^\alpha \tau_\alpha \equiv 0 \Leftrightarrow \exists U_I{}^\alpha , U_I{}^A :$$
$$\Lambda_I{}^i \partial_i S + \Lambda_I{}^\alpha \tau_\alpha \equiv U_I{}^\alpha \left(\Gamma_\alpha{}^i \partial_i S + \tau_\alpha\right)$$
$$+ U_I{}^A Z_A{}^\alpha \tau_\alpha . \tag{14}$$

Hence, the generators $\Lambda_I{}^i$, $\Lambda_I{}^\alpha$ of any identity between the equations of the system (8) reduce to the linear combinations of the generators Γ and Z modulo trivial generators:

$$\Lambda_I{}^i = U_I{}^\alpha \Gamma_\alpha{}^i + E_I{}^{ij}\partial_j S + E_I{}^{i\alpha}\tau_\alpha ,$$
$$\Lambda_I{}^\alpha = U_I{}^\alpha + U_I{}^A Z_A{}^\alpha - E_I{}^{i\alpha}\partial_i S + E_I{}^{\alpha\beta}\tau_\beta ,$$
$$E_I{}^{ij} = -E_I{}^{ji} , \quad E_I{}^{\alpha\beta} = -E_I{}^{\beta\alpha} . \tag{15}$$

Relation (10) leads to the identities between the identities (9), (11), because certain combination of the identity generators is trivial.

Also notice that the set of the identity generators $Z_A{}^\alpha$ is over-complete (13). This leads to further identities between the identities (11). These second level identities are irreducible, as their generators $Z_{1A_1}{}^A$ are assumed independent. Any set of identities, being labeled by the condensed index I_1, between the identities of identities is supposed generated by $Z_{1A_1}{}^A$:

$$\Lambda_{I_1}{}^A Z_A{}^\alpha \equiv E_{I_1}{}^{\alpha\beta}\tau_\beta \Leftrightarrow \Lambda_{I_1}{}^A = U_{I_1}{}^{A_1} Z_{1A_1}{}^A ,$$
$$E_{I_1}{}^{\alpha\beta} = -E_{I_1}{}^{\beta\alpha} . \tag{16}$$

Even though original action has no gauge symmetry, the involutive closure (8) of Lagrangian equations, being a non-Lagrangian system, enjoys non-trivial gauge algebra as demonstrated above. The general idea of inclusion of the Stueckelberg fields is to cast this gauge algebra back into Lagrangian setup by introducing extra fields. Specifically, the equations of the involutively closed system (8) should be zero order in the Stueckelberg fields of the Lagrangian Stueckelberg equations, while the gauge identities (9), (11) should be zero order of Noether identities for the Stueckelberg action. These reasons lead one to introduce the Stueckelberg

field ξ^α for every consequence τ_α included into involutive closure (8), while every gauge identity (9), (11) is assigned with the gauge parameter $\epsilon^\alpha, \epsilon^A$. Given the set of Stueckelberg fields and gauge parameters, we seek for the Stueckelberg action, and its gauge symmetry, as the power series in ξ:

$$\mathcal{S}_{St}(\phi, \xi) = \sum_{k=0}\mathcal{S}_k , \quad \mathcal{S}_0(\phi) = S(\phi) ,$$
$$\mathcal{S}_k(\phi, \xi) = W_{\alpha_1 \ldots \alpha_k}(\phi)\xi^{\alpha_1}\ldots\xi^{\alpha_k} , \quad k > 0 , \tag{17}$$

where the first order is defined by the differential consequences (6)

$$W_\alpha(\phi) = \left.\frac{\partial \mathcal{S}_{St}(\phi, \xi)}{\partial \xi^\alpha}\right|_{\xi=0} = \tau_\alpha . \tag{18}$$

Once the gauge identities (9), (11) are to be converted into the Noether identities of the action (17), corresponding gauge parameters ϵ^α and ϵ^A are introduced,

$$\delta_\epsilon \phi^i = R^i{}_\alpha(\phi, \xi)\epsilon^\alpha + R^i{}_A(\phi, \xi)\epsilon^A ,$$
$$\delta_\epsilon \xi^\alpha = R^\alpha{}_\beta(\phi, \xi)\epsilon^\beta + R^\alpha{}_A(\phi, \xi)\epsilon^A . \tag{19}$$

The gauge symmetry of the Stueckelberg action is equivalent to the Noether identities between the equations,

$$\delta_\epsilon \mathcal{S}_{St} = 0 , \quad \forall \epsilon^\alpha, \epsilon^A . \tag{20}$$

Let us expand the action (17) and gauge generators (19) in the Stueckelberg fields ξ, and substitute the expansions into the Noether identities. Comparing the identities (20) in zero order w.r.t. ξ with the identities (9), (11), we find zero order of the Stueckelberg gauge transformations,

$$\delta_\epsilon \phi^i = \Gamma^i{}_\alpha(\phi)\epsilon^\alpha + \ldots ,$$
$$\delta_\epsilon \xi^\alpha = \epsilon^\alpha + Z^\alpha{}_A(\phi)\epsilon^A + \ldots , \tag{21}$$

where $\Gamma^i{}_\alpha$ are the generators of consequences of Lagrangian equations (6) included into the involutive closure of original system, and $Z^\alpha{}_A$ are the generators of the identities (11) between τ_α. The dots stand for the ξ-depending terms. The generators $\Gamma^i{}_\alpha$ of the consequences (6) are reducible in the sense of relations (10). This results in the reducibility of the gauge identities (9), (11). Hence, the Noether identities (20) of the Stueckelberg action should be reducible as they begin with the identities between the equations of the involutive closure. Reducibility of the Noether identities means the gauge symmetry of the gauge symmetry. Comparing zero order of identities (20) with corresponding identities in the system (8), we find the gauge transformations of gauge parameters in zero order w.r.t. Stueckelberg fields,

$$\delta_\omega \epsilon^\alpha = Z^\alpha{}_A(\phi)\omega^A + \ldots ,$$
$$\delta_\omega \epsilon^A = -\omega^A + Z_1{}^A{}_{A_1}(\phi)\omega^{A_1} + \ldots , \tag{22}$$

where dots stand for the ξ-depending terms, $Z^\alpha{}_A$ are the null-vectors for the generators of consequences (10), and $Z_1{}^A{}_{A_1}$ are the generators of reducibility for $Z^\alpha{}_A$, see (13). The gauge parameters of symmetry for symmetry are denoted ω^A and ω^{A_1}. The gauge identities of identities in the original system (8) are reducible again. At the level of Stueckelberg theory this leads to the gauge symmetry of the parameters ω from the transformation above. This symmetry of symmetry in zero order in ξ is generated by the same operators as in the corresponding identities of identities of the original system. Hence, the next level gauge symmetry reads

$$\delta_\eta \omega^A = Z_1{}^A{}_{A_1}(\phi)\eta^{A_1} + \ldots , \quad \delta_\eta \omega^{A_1} = \eta^{A_1} + \ldots . \tag{23}$$

The second order in ξ of the Stueckelberg action (17), and the first order of the gauge transformations (19), can be found from Noether identities (20) at the first order, given the previous order (18), (21). Once the previous order is found, it is substituted into the expansion of the Noether identity up to the next order. This allows one to find the next order, etc. In this way, all the orders of the action and gauge generators are iteratively found. Up to the second order in ξ, the Stueckelberg action reads

$$\mathcal{S}_{St} = S(\phi) + \tau_\alpha(\phi)\xi^\alpha + \tfrac{1}{2}W_{\alpha\beta}(\phi)\xi^\alpha\xi^\beta + \dots,$$
$$W_{\alpha\beta} = W_{\beta\alpha}, \ W_{\alpha\beta} \approx \Gamma^i_\alpha \Gamma^j_\beta \partial^2_{ij} S. \tag{24}$$

The similar procedure applies to iteratively solving order by order the identities for identities proceeding from zero order (22), (23).

Given the regularity of the gauge algebra of the involutive closure (8) described at the beginning of this section, no obstructions can arise to the iterative inclusion of the Stueckelberg fields at any order. This can be proven by the tools of homological perturbation theory as described for the irreducible case in the article [8]. The main distinction of this proof from the usual homological perturbation theory procedures of gauge theories [12] is the unusual grading, where positive resolution degree is assigned to the Stueckelberg fields and their anti-fields, unlike the other fields. The aspect of reducibility can be accounted for in the homological perturbation theory with this grading in a natural way. This issue will be addressed elsewhere. From the point of view of the application in specific models, only the fact is important that the described procedure for including the Stueckelberg fields is unobstructed at all iteration steps.

3. Reducible Stueckelberg symmetry and dual formulation for massive spin 1

In this section, we exemplify the general method of inclusion of Stueckelberg fields with reducible gauge symmetry by the case of Proca model. The usual Stueckelberg scalar corresponds to the completion of the Proca system by the first order consequence — transversality condition. This is sufficient to make the Proca system involutive. However, the system can be completed also by the third order consequences, and it remains involutive. This option of the third order involutive closure, being treated by the procedure of previous section, leads to inclusion of the antisymmetric second rank tensor as the Stueckelberg field. The third order consequences turn out obeying the gauge identities of their own (cf. (11)), so we arrive at reducible Stueckelberg symmetry. This is no surprise once the antisymmetric tensor is introduced. The Stueckelberg action includes four derivatives, while the theory remains equivalent to the original Proca system. Besides exemplifying the general method, this case may have some interest of its own, as it demonstrates the scheme for constructing dual formulations for the fields of the same spin.

Consider the Proca Lagrangian for massive vector field A_μ in $d = 4$ Minkowski space,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu, \ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{25}$$

The Proca equations

$$\frac{\delta S}{\delta A^\mu} \equiv \Box A_\mu - \partial_\mu \partial^\nu A_\nu + m^2 A_\mu = 0 \tag{26}$$

are not involutive as such, as they admit the first order differential consequence

$$\tau \equiv \partial^\mu \frac{\delta S}{\delta A^\mu} = m^2 \partial^\mu A_\mu. \tag{27}$$

The system (26), (27) is involutive, so it can serve as the starting point for inclusion Stueckelberg fields. Once the consequence (27) is a scalar, corresponding Stueckelberg field should be scalar. The procedure of previous section for inclusion of Stueckelberg fields, being applied to the system (26), (27), reproduces the usual Stueckelberg formulation for the massive spin 1. The system (26), (27) can be complemented by the third order consequences, and still remains involutive. Consider the differential consequences

$$\tau_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu \delta^\rho_\nu - \partial_\nu \delta^\rho_\mu)\frac{\delta S}{\delta A^\rho} = \frac{1}{2}(\Box + m^2)F_{\mu\nu}. \tag{28}$$

These equations mean that the strength tensor of original field A_μ obeys Klein-Gordon equation. The system (26), (27), (28), being equivalent to the original Proca equations, is also involutive. So, it can be another starting point for including Stueckelberg fields. Following the general scheme of the previous section, let us list the identities between the equations of the involutive system (26), (27), (28). First, there are the identities (9) that follow from the definitions of the consequences. For the involutive closure of Proca system (26), (27), (28), these identities read

$$-\partial^\mu \frac{\delta S}{\delta A^\mu} + \tau = 0; \tag{29}$$

$$-\frac{1}{2}(\partial_\mu \delta^\rho_\nu - \partial_\nu \delta^\rho_\mu)\frac{\delta S}{\delta A^\rho} + \tau_{\mu\nu} = 0. \tag{30}$$

The consequences $\tau_{\mu\nu}$ (28) are reducible in the sense of identities (11). These identities read

$$\varepsilon^{\mu\nu\rho\lambda}\partial_\nu \tau_{\rho\lambda} = 0, \tag{31}$$

where $\varepsilon^{\mu\nu\rho\lambda}$ is Levi-Chivita symbol. These identities are reducible in their own turn, as the divergence of the l.h.s. identically vanishes for any $\tau_{\mu\nu}$. It is the second level identity (cf. (13)):

$$\partial_\mu \varepsilon^{\mu\nu\rho\lambda}\partial_\nu = 0. \tag{32}$$

The identities (29), (30) between the equations of the third order involutive closure of Proca system and the identities of identities (31), (32) allow one to identify all the ingredients needed for inclusion of Stueckelberg fields with reducible gauge symmetry: the generators of consequences Γ (6), null-vectors Z of Γ's (cf. (10)), and null-vectors of the null-vectors Z_1 (cf. (13)):

$$\Gamma^\mu = -\partial^\mu, \ \Gamma_{\mu\nu}{}^\rho = -\frac{1}{2}(\partial_\mu \delta^\rho_\nu - \partial_\nu \delta^\rho_\mu),$$
$$Z^{\mu\lambda\rho} = \varepsilon^{\mu\nu\lambda\rho}\partial_\nu, \ Z_{1\mu} = \partial_\mu. \tag{33}$$

With all the ingredients at hands, following the general procedure of Section 2, we iteratively construct the Stueckelberg action, generators of gauge symmetries, and symmetries of symmetries. Once the original action is quadratic and the identity generators (33) are field-independent, the procedure terminates at the first iteration. The Stueckelberg action and reducible gauge symmetry transformations read

$$\mathcal{S}_{St} = \int d^4 x \Big(-\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} - \frac{1}{2}\partial_\mu \partial^\rho B_{\nu\rho}\big(\partial^\mu \partial_\lambda B^{\nu\lambda}$$

$$+ 2\,\partial^\mu A^\nu\big) + \frac{m^2}{2}\big(A_\mu A^\mu + \partial_\mu \varphi\,\partial^\mu \varphi + \partial^\nu B_{\mu\nu}\partial_\rho B^{\mu\rho}\big) \tag{34}$$

$$+ m^2 A_\mu \big(\partial^\mu \varphi + \partial_\nu B^{\mu\nu}\big)\Big),$$

$$\delta_\epsilon A^\mu = -\partial^\mu \epsilon - \partial_\nu \epsilon^{\mu\nu}, \quad \delta_\epsilon \varphi = \epsilon,$$
$$\delta_\epsilon B^{\mu\nu} = \epsilon^{\mu\nu} + \varepsilon^{\mu\nu\rho\lambda}\partial_\rho \epsilon_\lambda, \tag{35}$$

where φ and $B^{\mu\nu} = -B^{\nu\mu}$ are the Stueckelberg fields corresponding to the consequences τ and $\tau_{\mu\nu}$ (27), (28), while ϵ, ϵ_μ, $\epsilon_{\mu\nu}$ are the gauge parameters corresponding to the gauge identities (29), (30), (31). By direct computation, one can easily see that action (34) enjoys symmetry (35) indeed. Given the null-vectors Z and Z_1 (33), the symmetries of symmetries are constructed following the general prescription (22), (23),

$$\delta_\omega \epsilon = 0, \quad \delta_\omega \epsilon_\mu = -\omega_\mu - \partial_\mu \omega,$$
$$\delta_\omega \epsilon^{\mu\nu} = \varepsilon^{\mu\nu\rho\lambda} \partial_\rho \omega_\lambda, \tag{36}$$

$$\delta_\eta \omega = \eta, \quad \delta_\eta \omega_\mu = -\partial_\mu \eta, \tag{37}$$

where (36) are the gauge symmetry transformations of the original gauge parameters ϵ, while (37) is the gauge symmetry of the second level gauge parameters ω.

Consider the Lagrangian equations for Stueckelberg action (25),

$$\frac{\delta \mathcal{S}_{St}}{\delta A^\mu} \equiv \Box A_\mu - \partial_\mu \partial^\nu A_\nu + \Box \partial^\nu B_{\mu\nu}$$

$$+ m^2 A_\mu + m^2 \partial_\mu \varphi + m^2 \partial^\nu B_{\mu\nu} = 0, \tag{38}$$

$$\frac{\delta \mathcal{S}_{St}}{\delta \varphi} \equiv -m^2 (\Box \varphi + \partial^\mu A_\mu) = 0, \tag{39}$$

$$\frac{\delta \mathcal{S}_{St}}{\delta B^{\mu\nu}} \equiv \frac{1}{2} (\Box + m^2) (\partial_\mu \partial^\rho B_{\rho\nu} - \partial_\nu \partial^\rho B_{\rho\mu}$$

$$+ \partial_\mu A_\nu - \partial_\nu A_\mu) = 0. \tag{40}$$

These equations involve the fourth order derivatives, so equivalence with the original Proca theory may seem doubtful. However, these equations enjoy the reducible gauge symmetry (35). This symmetry admits gauge fixing conditions

$$\varphi = 0, \quad B_{\mu\nu} = 0. \tag{41}$$

This gauge eliminates all the Stueckelberg fields and reduces the system to Proca equations (26).

It is interesting to notice another admissible gauge fixing for the symmetry (35):

$$\varphi = 0, \quad A_\mu = 0, \quad \varepsilon_{\mu\nu\rho\lambda} \partial^\nu B^{\rho\lambda} = 0. \tag{42}$$

As this gauge fixing kills scalar φ and vector field A_μ, equations (38)-(39) reduce to third-order equation (1), while (40) becomes its differential consequence. Let us detail fixing of the gauge parameters by conditions (42). Taking variation of (42) we arrive at the conditions

$$\delta_\epsilon \varphi = \epsilon = 0, \quad \delta_\epsilon A_\mu = \partial_\mu \epsilon + \partial^\nu \epsilon_{\mu\nu} = 0,$$
$$\varepsilon_{\mu\nu\rho\lambda} \partial^\nu \delta_\epsilon B^{\rho\lambda} = \varepsilon_{\mu\nu\rho\lambda} \partial^\nu \epsilon^{\rho\lambda} - \Box \epsilon_\mu + \partial_\mu \partial^\nu \epsilon_\nu = 0. \tag{43}$$

So, the gauge conditions (42) restrict the gauge parameters by the relations

$$\epsilon = 0, \quad \partial_\mu \epsilon^{\mu\nu} = 0,$$
$$\varepsilon_{\mu\nu\rho\lambda} \partial^\nu \epsilon^{\rho\lambda} - \Box \epsilon_\mu + \partial_\mu \partial^\nu \epsilon_\nu = 0. \tag{44}$$

Once $\epsilon = 0$, the second of these equations means $\epsilon^{\mu\nu} = \varepsilon^{\mu\nu\rho\lambda} \partial_\rho \omega_\lambda$, where ω_λ is arbitrary. Substituting that into the last relation we see that the difference between the gauge parameter ϵ_μ and ω_μ obeys free Maxwell equations. Maxwell equations have unique solution modulo the gradient of arbitrary scalar $\partial_\mu \omega$, given the Cauchy data. So, the general solution of equations (44) reads

$$\epsilon = 0, \quad \epsilon_\mu = \omega_\mu + \partial_\mu \omega, \quad \epsilon^{\mu\nu} = \varepsilon^{\mu\nu\rho\lambda} \partial_\rho \omega_\lambda, \tag{45}$$

where ω_μ, ω are arbitrary functions. This means, the gauge conditions (42) fix parameters ϵ, ϵ_μ, $\epsilon_{\mu\nu}$ modulo symmetry of symmetry (36). The ambiguity of this type always remains unfixed at the level of field equations for original fields in the case of reducible gauge symmetry. In the BRST formalism, this ambiguity is fixed by imposing gauge conditions on the ghosts and introducing ghosts for ghosts [12].

Admissibility of the gauge fixing condition such that kills the original vector field means that A_μ can be considered as a pure gauge from the viewpoint of the action (25) with gauge symmetry (35). This is true indeed, given the transformation $\delta_\epsilon A_\mu$ (35) which demonstrates that both gradient and transverse parts are ambiguous of the vector A_μ, so only zero modes can survive in the gauge transformations. Once transformations for the fields A_μ and $B^{\mu\nu}$ share the same gauge parameter $\epsilon^{\mu\nu}$, the gauge ambiguity can be equally well fixed either by the conditions killing A and residual ambiguity in B, or by fixing B.

The equations (1), being one of the gauge fixed forms of the Stueckelberg system (38), (39), (40) are equivalent to the original Proca system. These third order non-Lagrangian equations can be considered a dual form of the vector representation (2) of massive spin 1 particle, as it has been already explained in the Introduction. One can switch between these dual forms by imposing different gauges in the same Lagrangian theory. This example demonstrates that if the inclusion of Stueckelberg fields begins with the higher order involutive closure of the original theory, the Stueckelberg action, being equivalent to the original non-involutive theory, can include dual formulations of the same irreps. This topic is further discussed in Conclusion.

4. Conclusion

Let us summarize and discuss the results. First, we propose a systematic way for inclusion of Stueckelberg fields such that guarantees equivalence of the resulting gauge theory to the original system. The starting point for inclusion of Stueckelberg fields is the involutive closure of original Lagrangian equations (8). If the closure includes an over-complete set of consequences (see (11)), the Stueckelberg symmetry turns out reducible. In any case, the Stueckelberg theory is iteratively constructed for any involutive closure of Lagrangian equations without obstructions at any stage, be the consequences (6) reducible or not. In this sense, the covariant method is a complete analogue to the Hamiltonian method of conversion of the second class constraints into the first class ones.

The interesting option for inclusion of Stueckelberg fields is to start with the involutive closure of the higher order than it is minimally sufficient. This option is exemplified in Section 3 by the third order involutive closure of Proca model, where the added consequences are reducible. Following the general procedure of inclusion of the Stueckelberg fields, we arrive to the higher derivative Stueckelberg action (34) which is equivalent to the first derivative Proca action. This Stueckelberg model for massive spin 1 turns out comprising two dual field theoretical realizations for the same irreducible representation. The first one is the original Proca model, and the second one is the third order formulation (1) in terms of the antisymmetric tensor field. The field $B^{\mu\nu}$ can be considered as a potential for the original transverse vector (cf. (3)). Notice that various dual formulations are studied once and again for the same spin representation. For the most recent results on this topic and further references we refer to the article [13]. Important motivation for studying dual formulations is that they are inequivalent, in general, w.r.t. inclusion of interactions. Among the examples of this sort, we can mention the representation of the massless spin 2 by the third rank tensor field with Young diagram of the hook type [14]. Unlike the representation of the same spin by the symmetric second rank tensor, the hook does not admit inclusion of con-

sistent interactions [15]. Similar phenomena are observed among the higher spin gravities. In particular, the long known light-cone analysis of the higher spin vertices in Minkowski space [16], [17] demonstrates admissibility of the interactions such that are missing among the deformations of Fronsdal's Lagrangians for symmetric tensors. There is a growing evidence that Lagrangians for dual formulations of higher spins can admit these vertices. For the recent results, discussion of the area, and further references we refer to [18], [19]. Notice that the considered dual formulations are typically connected to each other algebraically, hence all the actions are of the same order. Proposed scheme of inclusion Stueckelberg fields proceeds from the involutive closure of the original Lagrangian equations. If the starting point is the higher order closure of the original system, corresponding Stueckelberg field, being candidate for the dual to the original field, would be involved in the Lagrangian with higher derivatives. This dual would be connected to the original field by a differential relation, like a potential (cf. (3)). So, this scenario of inclusion Stueckelberg fields can serve as a tool for constructing a different type of dual formulations. For example, if the original fields are symmetric, the second order Lagrangian equations can admit the third order involutive closure with differential consequences, being the tensors of hook type. Corresponding higher derivative Stueckelberg Lagrangian has to be equivalent to the original one by construction, while the hook tensors would serve as dual to the original fields, following the pattern of Section 3. These dual models can have their chances for consistent interactions as the potentials can be less obstructive to deformations than corresponding strength tensors.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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