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EQUATIONS OVER DIRECT POWERS OF ALGEBRAIC STRUCTURES IN RELATIONAL LANGUAGES[1]

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For a semigroup S (group G) we study relational equations and describe all semigroups S with equationally Noetherian direct powers. It follows that any group G has equationally Noetherian direct powers if we consider G as an algebraic structure of a certain relational language. Further we specify the results as follows: if a direct power of a finite semigroup S is equationally Noetherian, then the minimal ideal $\mathrm{Ker}(S)$ of S is a rectangular band of groups and $\mathrm{Ker}(S)$ coincides with the set of all reducible elements.

Keywords: *relations, groups, semigroups, direct powers, equationally Noetherian algebraic structures.*

Introduction

Let $\mathcal{A}$ be an algebraic structure of a functional language $\mathcal{L}$ with a universe A. In other words, there are certain functions and constants over $\mathcal{A}$ that correspond to symbols of $\mathcal{L}$. One can define a structure $\mathrm{Pr}(\mathcal{A})$ with the universe A of a pure relational language $\mathcal{L}_{\mathrm{pred}}$ as follows:

$$R_f(x_1,\ldots,x_n,y)=\{(x_1,\ldots,x_n,y): f(x_1,\ldots,x_n)=y\in\mathcal{A}\};$$
$$R_c(x)=\{x : x=c\in\mathcal{A}\},$$

where functional and constant symbols f,c belong to the language $\mathcal{L}$. Namely, the relation $R_f\in\mathcal{L}_{\mathrm{pred}}$ ($R_c\in\mathcal{L}_{\mathrm{pred}}$) is the graph of a function f (respectively, constant c).

The $\mathcal{L}_{\mathrm{pred}}$-structure $\mathrm{Pr}(\mathcal{A})$ is called the *predicatization* of an $\mathcal{L}$-structure $\mathcal{A}$. In particular, if $\mathcal{A}$ is a group of the language $\mathcal{L}_g=\{\cdot,^{-1},1\}$, then $\mathrm{Pr}(\mathcal{A})$ is an algebraic structure of the language $\mathcal{L}_{g-\mathrm{pred}}$ with the following relations:

$$M(x,y,z)\Leftrightarrow xy=z; \tag{1}$$
$$I(x,y)\Leftrightarrow x=y^{-1}; \tag{2}$$
$$E(x)\Leftrightarrow x=1. \tag{3}$$

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Notice that any equation over a group $\mathcal{A}$ may be rewritten in the language $\mathcal{L}_{g-\mathrm{pred}}$ by the introducing new variables. For example, the equation $x^{-1}y^{-1}xy = 1$ has the following correspondence in the relational language $\mathcal{L}_{g-\mathrm{pred}}$:

$$\mathrm{Pr}(\mathbf{S}) = \begin{cases} I(x, x_1), \\ I(y, y_1), \\ M(x_1, y_1, z_1), \\ M(z_1, x, z_2), \\ M(z_2, y, z_3), \\ E(z_3). \end{cases}$$

It is easy to see that the projection of the solution set of $\mathbf{S}$ onto the variables x, y gives the solution set of the initial equation $x^{-1}y^{-1}xy = 1$. More generally, for any finite set of group equations $\mathbf{S}$ in variables X there exists a system $\mathrm{Pr}(\mathbf{S})$ of equations in the language $\mathcal{L}_{g-\mathrm{pred}}$ such that the solution set of $\mathbf{S}$ is the projection of the solution set $\mathrm{Pr}(\mathbf{S})$ onto the variables X. Hence, there arises the following important problem.

Problem. What properties of a finite system $\mathbf{S}$ are determined by the system $\mathrm{Pr}(\mathbf{S})$?

This problem was originally studied in [1], where it was proved the general results for relational structures $\mathrm{Pr}(\mathbf{S})$.

We study equations over direct products of semigroups. Namely, for a finite semigroup S we give necessary and sufficient condition whether the direct power $\Pi\,\mathrm{Pr}(S)$ is equationally Noetherian. It continues the research [2], where we found the necessary and sufficient conditions for the equationally Noetherian property of direct powers of functional algebraic structures (groups, rings, monoids). For example, a group (ring) has equationally Noetherian direct powers in a functional language with constants iff it is abelian (respectively, with zero multiplication).

On the other hand, we prove below that any finite group in the language $\mathcal{L}_{g-\mathrm{pred}}$ has equationally Noetherian direct powers (Corollary 1). Moreover, the similar result holds for the natural generalizations of groups: quasi-groups and loops (Remark 1).

However, the class of semigroups has a nontrivial classification in the relational language. We find two quasi-identities

$$\forall a \forall b \forall \alpha \forall \beta \left((a\alpha = a\beta) \to (b\alpha = b\beta) \right); \tag{4}$$

$$\forall a \forall b \forall \alpha \forall \beta \left((\alpha a = \beta a) \to (\alpha b = \beta b) \right) \tag{5}$$

such that a finite semigroup S satisfies (4), (5) iff any direct power of $\mathrm{Pr}(S)$ is equationally Noetherian (Theorem 1).

In the class of finite semigroups the conditions (4), (5) imply that the minimal ideal (kernel) of a semigroup S is a rectangular band of groups, and the kernel $\mathrm{Ker}(S)$ (the minimal ideal of S) coincides with the ideal of reducible elements of S. If the kernel of a finite semigroup S is a group, then the converse statement also holds (Theorem 4). However, the converse statement is not true in general (Example 1).

1. Basic notions

An algebraic structure of the language $\mathcal{L}_{s-\mathrm{pred}} = \{M^{(3)}\}$ ($\mathcal{L}_{g-\mathrm{pred}} = \{M^{(3)}, I^{(2)}, E^{(1)}\}$) is called the *predicatization* of a semigroup S (group G) if the operations over S (G) corresponds to the relations (1)–(3). The predicatization of a semigroup S (group G) is denoted by $\mathrm{Pr}(S)$ (respectively, $\mathrm{Pr}(G)$).

Following [3], we give the main definitions of algebraic geometry over algebraic structures (below $\mathcal{L} \in \{\mathcal{L}_{s-\mathrm{pred}}, \mathcal{L}_{p-\mathrm{pred}}\}$).

An *equation over* $\mathcal{L}$ *($\mathcal{L}$-equation)* is an atomic formula over $\mathcal{L}$. The examples of equations are the following: $M(x,x,x)$, $M(x,y,x)$ ($\mathcal{L}_{s-\mathrm{pred}}$-equations); $M(x,x,y)$, $I(x,y)$, $I(x,x)$, $E(x)$ ($\mathcal{L}_{g-\mathrm{pred}}$-equations).

A *system of* $\mathcal{L}$*-equations* ($\mathcal{L}$*-system* for shortness) is an arbitrary set of $\mathcal{L}$-equations. Notice that we will consider only systems in a finite set of variables $X = \{x_1, x_2, \dots, x_n\}$. The set of all solutions of $\mathbf{S}$ in an $\mathcal{L}$-structure $\mathcal{A}$ is denoted by $\mathrm{V}_{\mathcal{A}}(\mathbf{S}) \subseteq \mathcal{A}^n$. A set $Y \subseteq \mathcal{A}^n$ is said to be an *algebraic set over* $\mathcal{A}$ if there exists an $\mathcal{L}$-system $\mathbf{S}$ with $Y = \mathrm{V}_{\mathcal{A}}(\mathbf{S})$. If the solution set of an $\mathcal{L}$-system $\mathbf{S}$ is empty, $\mathbf{S}$ is said to be *inconsistent*. Two $\mathcal{L}$-systems $\mathbf{S}_1, \mathbf{S}_2$ are called *equivalent over an* $\mathcal{L}$*-structure* $\mathcal{A}$ if $\mathrm{V}_{\mathcal{A}}(\mathbf{S}_1) = \mathrm{V}_{\mathcal{A}}(\mathbf{S}_2)$.

An $\mathcal{L}$-structure $\mathcal{A}$ is $\mathcal{L}$*-equationally Noetherian* if any infinite $\mathcal{L}$-system $\mathbf{S}$ is equivalent over $\mathcal{A}$ to a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure. By $\mathcal{L}(\mathcal{A})$ we denote the language $\mathcal{L} \cup \{a : a \in \mathcal{A}\}$ extended by new constants symbols which correspond to elements of $\mathcal{A}$. The language extension allows us to use constants in equations. The examples of equations in the extended languages are the following: $M(x,y,a)$ ($\mathcal{L}_{s-\mathrm{pred}}(S)$-equation and $a \in S$); $M(a,x,b)$, $I(x,a)$, $E(a)$ ($\mathcal{L}_{g-\mathrm{pred}}(S)$-equations and $a, b \in G$). Obviously, the class of $\mathcal{L}(\mathcal{A})$-equations is wider than the class of $\mathcal{L}$-equations, so an $\mathcal{L}$-equationally Noetherian algebraic structure $\mathcal{A}$ may lose this property in the language $\mathcal{L}(\mathcal{A})$.

One can directly prove that any finite $\mathcal{L}(\mathcal{A})$-structure $\mathcal{A}$ is always $\mathcal{L}(\mathcal{A})$-equationally Noetherian.

Since the algebraic structures $\mathcal{A}$ and $\mathrm{Pr}(\mathcal{A})$ have the same universe, we will write below $\mathrm{V}_{\mathcal{A}}(\mathbf{S})$ ($\mathcal{L}(\mathcal{A})$) instead of $\mathrm{V}_{\mathrm{Pr}(\mathcal{A})}(\mathbf{S})$ (respectively, $\mathcal{L}(\mathrm{Pr}(\mathcal{A}))$).

Let $\mathcal{A}$ be a relational $\mathcal{L}$-structure. The *direct power* $\Pi\mathcal{A} = \prod\limits_{i \in I} \mathcal{A}$ of $\mathcal{A}$ is the set of all sequences $[a_i : i \in I]$ and any relation $R \in \mathcal{L}$ is defined as follows

$$R\big([a_i^{(1)} : i \in I], [a_i^{(2)} : i \in I], \dots, [a_i^{(n)} : i \in I]\big) \Leftrightarrow R\big(a_i^{(1)}, a_i^{(2)}, \dots, a_i^{(n)}\big) \text{ for each } i \in I.$$

A map $\pi_k \colon \Pi\mathcal{A} \to \mathcal{A}$ is called the *projection onto the i-th coordinate* if $\pi_k([a_i : i \in I]) = a_k$.

Let $E(X)$ be an $\mathcal{L}(\Pi\mathcal{A})$-equation over a direct power $\Pi\mathcal{A}$. We may rewrite $E(X)$ in the form $E(X, \overrightarrow{\mathbf{C}})$, where $\overrightarrow{\mathbf{C}}$ is an array of constants occurring in the equation $E(X)$. One can introduce the *projection of an equation* onto the i-th coordinate as follows:

$$\pi_i(E(X)) = \pi_i(E(X, \overrightarrow{\mathbf{C}})) = E(X, \pi_i(\overrightarrow{\mathbf{C}})),$$

where $\pi_i(\overrightarrow{\mathbf{C}})$ is an array of the i-th coordinates of the elements from $\overrightarrow{\mathbf{C}}$. For example, the $\mathcal{L}_{s-\mathrm{pred}}(\Pi\mathcal{A})$-equation $M(x, [a_1, a_2, a_3, \dots], [b_1, b_2, b_3, \dots])$ has the following projections

$$\begin{gathered} M(x, a_1, b_1), \\ M(x, a_2, b_2), \\ M(x, a_3, b_3), \\ \dots \end{gathered}$$

Obviously, any projection of an $\mathcal{L}(\Pi\mathcal{A})$-equation is an $\mathcal{L}(\mathcal{A})$-equation.

Let us take an $\mathcal{L}(\Pi\mathcal{A})$-system $\mathbf{S} = \{E_j(X) : j \in J\}$. The i-th projection of $\mathbf{S}$ is the $\mathcal{L}(\mathcal{A})$-system defined by $\pi_i(\mathbf{S}) = \{\pi_i(E_j(X)) : j \in J\}$. The projections of an $\mathcal{L}(\Pi\mathcal{A})$-system $\mathbf{S}$ allow to describe the solution set of $\mathbf{S}$ by

$$\mathrm{V}_{\Pi\mathcal{A}}(\mathbf{S}) = \{[P_i : i \in I] : P_i \in \mathrm{V}_{\mathcal{A}}(\pi_i(\mathbf{S}))\}. \tag{6}$$

In particular, if one of the projections $\pi_i(\mathbf{S})$ is inconsistent, so is $\mathbf{S}$.

The following statement immediately follows from the description (6) of the solution set over a direct powers.

Lemma 1. Let $\mathbf{S} = \{E_j(X) : j \in J\}$ be an $\mathcal{L}(\Pi\mathcal{A})$-system over $\Pi\mathcal{A}$. If one of the projections $\pi_i(\mathbf{S})$ is inconsistent, so is $\mathbf{S}$. Moreover, if $\mathcal{A}$ is $\mathcal{L}(\mathcal{A})$-equationally Noetherian, then an inconsistent $\mathcal{L}(\Pi\mathcal{A})$-system $\mathbf{S}$ is equivalent to a finite subsystem.

Proof. The first assertion directly follows from (6). Suppose $\mathcal{A}$ is $\mathcal{L}$-equationally Noetherian, and $\pi_i(\mathbf{S})$ is inconsistent. Hence, $\pi_i(\mathbf{S})$ is equivalent to its finite inconsistent subsystem $\{\pi_i(E_j(X)) : j \in J'\}$, $|J'| < \infty$, and the finite subsystem $\mathbf{S}' = \{E_j(X) : j \in \in J'\} \subseteq \mathbf{S}$ is also inconsistent. ∎

2. Predicatization of semigroups and groups

Theorem 1. Let $\mathrm{Pr}(S)$ be the predicatization of a finite semigroup S. A direct power of $\mathrm{Pr}(S)$ is $\mathcal{L}_{s-\mathrm{pred}}(\Pi S)$-equationally Noetherian iff the quasi-identities (4), (5) hold in S.

Proof. First, we prove the "if" part of the theorem. Suppose S satisfies (4,5) and consider an infinite $\mathcal{L}_{s-\mathrm{pred}}(\Pi S)$-system $\mathbf{S}$. One can represent $\mathbf{S}$ as a finite union of the following systems

$$\mathbf{S} = \bigcup_{1\leqslant i,j\leqslant n} \mathbf{S}_{cij} \bigcup_{1\leqslant i,j\leqslant n} \mathbf{S}_{icj} \bigcup_{1\leqslant i,j\leqslant n} \mathbf{S}_{ijc} \bigcup_{1\leqslant i\leqslant n} \mathbf{S}_{cci} \bigcup_{1\leqslant i\leqslant n} \mathbf{S}_{cic} \bigcup_{1\leqslant i\leqslant n} \mathbf{S}_{icc} \bigcup \mathbf{S}_0, \tag{7}$$

where each equation of $\mathbf{S}_0$ is one of the following types:

1) $x_i = x_j$;
2) $x_i = \mathbf{c}_j$;
3) $\mathbf{c}_i = \mathbf{c}_j$;
4) $M(x_i, x_j, x_k)$;
5) $M(\mathbf{c}_i, \mathbf{c}_j, \mathbf{c}_k)$,

and $\mathbf{S}_{cij} = \{M(\mathbf{c}_k, x_i, x_j) : k \in K\}$, $\mathbf{S}_{icj} = \{M(x_i, \mathbf{c}_k, x_j) : k \in K\}$, $\mathbf{S}_{ijc} = \{M(x_i, x_j, \mathbf{c}_k) : k \in K\}$, $\mathbf{S}_{cci} = \{M(\mathbf{c}_k, \mathbf{d}_k, x_i) : k \in K\}$, $\mathbf{S}_{cic} = \{M(\mathbf{c}_k, x_i, \mathbf{d}_k) : k \in K\}$, $\mathbf{S}_{icc} = = \{M(x_i, \mathbf{c}_k, \mathbf{d}_k) : k \in K\}$ $(\mathbf{c}_k, \mathbf{d}_k \in \Pi\,\mathrm{Pr}(S))$, where each system above has its own index set K.

Clearly, the system $\mathbf{S}_0$ is equivalent to its finite subsystem. So it is sufficient to prove that the other systems are equivalent to their finite subsystems. According to Lemma 1, we may assume that all systems below are consistent.

Thus, we have the following cases:

1) Let $\mathbf{S}_{icc} = \{M(x_i, \mathbf{c}_k, \mathbf{d}_k) : k \in K\}$ and $M(x_i, \mathbf{c}_1, \mathbf{d}_1)$ be an arbitrary equation of $\mathbf{S}_{icc}$. Since $\mathbf{S}_{icc}$ is consistent, then one can choose $\bar{\alpha} \in \mathrm{V}_{\Pi S}(\mathbf{S}_{icc})$, $\bar{\beta} \in \mathrm{V}_{\Pi S s}(M(x_i, \mathbf{c}_1, \mathbf{d}_1))$. We have $\bar{\alpha}\mathbf{c}_1 = \bar{\beta}\mathbf{c}_1 = \mathbf{d}_1$. By the quasi-identities (4), (5), $\bar{\alpha}\mathbf{c}_k = \bar{\beta}\mathbf{c}_k$ for any $\mathbf{c}_k$. Hence, $\bar{\beta}$ satisfies all equations from $\mathbf{S}_{icc}$, Thus, $\mathbf{S}_{icc}$ is equivalent to the equation $M(x_i, \mathbf{c}_1, \mathbf{d}_1)$. The proof for the system $\mathbf{S}_{cic}$ is similar.
2) Let $\mathbf{S}_{cci} = \{M(\mathbf{c}_k, \mathbf{d}_k, x_i) : k \in K\}$. Since the system $\mathbf{S}_{cci}$ is consistent, the products $\mathbf{c}_k\mathbf{d}_k$ are equal to each other, hence $\mathbf{c} = \mathbf{c}_k\mathbf{d}_k$ for all $k \in K$. Thus, the whole system $\mathbf{S}_{cci}$ is equivalent to any equation $M(\mathbf{c}_k, \mathbf{d}_k, x_i)$.
3) Let $\mathbf{S}_{icj} = \{M(x_i, \mathbf{c}_k, x_j) : k \in K\}$ (the proof for $\mathbf{S}_{cij}$ is similar). Since $\mathbf{S}_{icj}$ is consistent, there exist a point $(\bar{\alpha}, \bar{\beta}) \in \mathrm{V}_{\Pi S}(\mathbf{S}_{icj})$ and the equalities $\bar{\alpha}\mathbf{c}_k = \bar{\alpha}\mathbf{c}_l = \bar{\beta}$ hold for any $k, l \in K$. By (4), (5), for any $\bar{\gamma} \in \Pi S$ it holds $\bar{\gamma}\mathbf{c}_k = \bar{\gamma}\mathbf{c}_l$. Thus, the solution set of $\mathbf{S}_{icj}$ is $Y = \{(\bar{\gamma}, \bar{\gamma}\mathbf{c}_{k_0}) \mid \bar{\gamma} \in \Pi S\}$ for a fixed $k_0 \in K$. Thus, $\mathbf{S}_{icj}$ is equivalent to the equation $M(x_i, \mathbf{c}_{k_0}, x_j)$.

4) Let $\mathbf{S}_{ijc} = \{M(x_i, x_j, \mathbf{c}_k) : k \in K\}$. Since the system $\mathbf{S}_{ijc}$ is consistent, the elements $\mathbf{c}_k$ ($k \in K$) are equal to each other. Hence, the system $\mathbf{S}_{ijc}$ consists of the same equations. Thus, $\mathbf{S}_{ijc}$ is equivalent to any equation $M(x_i, x_j, \mathbf{c}_k)$.

Now, we prove the "only if" part of the theorem. Suppose the quasi-identity (4) does not hold in S (for the formula (5) the proof is similar). It follows there exist elements a, b, α, β such that $a\alpha = a\beta = c$, $b\alpha \neq b\beta$. Let us consider the system

$$\mathbf{S} = \{M(\mathbf{a}_n, x, \mathbf{c}_n) : n \in \mathbb{N}\},$$

where

$$\mathbf{a}_n = [\underbrace{b, \ldots, b}_{n \text{ times}}, a, a, \ldots], \quad \mathbf{c}_n = [\underbrace{b\beta, \ldots, b\beta}_{n \text{ times}}, c, c, \ldots].$$

One can directly check that the point

$$\mathbf{a} = [\underbrace{\beta, \ldots, \beta}_{n \text{ times}}, \alpha, \alpha, \ldots]$$

satisfies the first n equations of $\mathbf{S}$ (since we obtain the true equalities $a\beta = c$ or $b\beta = b\beta$). However the $(n+1)$-th equation of $\mathbf{S}$ gives $\mathbf{a}_{n+1}\mathbf{a} \neq \mathbf{c}_{n+1}$, since its $(n+1)$-th projection defines the equation $bx = b\beta$, but $b\alpha \neq b\beta$. Thus, $\mathbf{S}$ is not equivalent to any finite subsystem. ■

Corollary 1. Let $\mathrm{Pr}(G)$ be the predicatization of a finite group G. Then any direct power of $\mathrm{Pr}(G)$ is $\mathcal{L}_{g-\mathrm{pred}}(\Pi G)$-equationally Noetherian.

Proof. Since the equality $a\alpha = a\beta$ ($\alpha a = \beta a$) implies $\alpha = \beta$ in any group, the quasi-identities (4), (5) obviously hold in G. Thus, any infinite system of the form $\{M(*, *, *) : i \in I\}$ is equivalent to a finite subsystem.

One can directly prove that for any finite group G the infinite systems of the form $\{I(*, *) : i \in I\}$ ($\{E(*) : i \in I\}$) are also equivalent to their finite subsystems over ΠG.

Thus, any system of $\mathcal{L}_{g-\mathrm{pred}}(\Pi G)$-equations is equivalent over ΠG to its finite subsystem. ■

Remark 1. The Corollary 1 also holds for finite quasi-groups. Notice that a quasi-group is a non-associative generalization of a group. Any quasi-group admits the analogue of divisibility, hence the quasi-identities (4), (5) obviously hold in any quasi-group. Thus, any direct power of a quasi-group G is $\mathcal{L}_{s-\mathrm{pred}}(\Pi G)$-equationally Noetherian (here we consider quasi-groups and loops in the language $\mathcal{L}_{s-\mathrm{pred}}$, since not any quasi-group admits the relations $I(x, y)$ and $E(x)$).

Below we study finite semigroups S that satisfy Theorem 1.

A subset $I \subseteq S$ is called a *left* (*right*) *ideal* if for any $s \in S$, $a \in I$ it holds $sa \in I$ ($as \in I$). An ideal which is right and left simultaneously is said to be *two-sided* (or an *ideal* for shortness).

A semigroup S with a unique ideal $I = S$ is called *simple*. Let us remind the classical Sushkevich — Rees theorem for finite simple semigroups.

Theorem 2. For any finite simple semigroup S there exist a finite group G and finite sets I, Λ such that S is isomorphic to the set of triples (λ, g, i), $g \in G$, $\lambda \in \Lambda$, $i \in I$. The multiplication over the triples (λ, g, i) is defined by

$$(\lambda, g, i)(\mu, h, j) = (\lambda, g p_{i\mu} h, j),$$

where $p_{i\mu} \in G$ is an element of a matrix $\mathbf{P}$ such that

1) $\mathbf{P}$ consists of $|I|$ rows and $|\Lambda|$ columns;

2) the elements of the first row and the first column equal $1 \in G$ (i.e., $\mathbf{P}$ is *normalized*).

Following Theorem 2, we denote any finite simple semigroup S by $S = (G, \mathbf{P}, \Lambda, I)$.

The minimal ideal of a semigroup S is called a *kernel* and denoted by $\mathrm{Ker}(S)$ (any finite semigroup always has a unique kernel, and the kernel is always simple, i.e., $\mathrm{Ker}(S)$ satisfies Theorem 2). Obviously, if $S = \mathrm{Ker}(S)$, then the semigroup is simple. If $\mathrm{Ker}(S)$ is a group, then S is said to be a *homogroup*. The next theorem contains the necessary information about homogroups.

Theorem 3 [4]. In a homogroup S the identity element e of the kernel $\mathrm{Ker}(S)$ is idempotent ($e^2 = e$) and belongs to the center of S (i.e., e commutes with any $s \in S$).

A semigroup S is called a *rectangular band of groups* if $S = (G, \mathbf{P}, \Lambda, I)$ and $p_{i\lambda} = 1$ for any $i \in I$, $\lambda \in \Lambda$.

Lemma 2. Suppose a finite simple semigroup S satisfies (4), (5). Then S is a rectangular band of groups.

Proof. By Theorem 2, $S = (G, \mathbf{P}, \Lambda, I)$ for some finite group G, matrix $\mathbf{P}$ and finite sets of indexes Λ, I.

Assume that $|\Lambda| > 1$ and $p_{i\lambda} \neq 1$ for some i, λ.

Let $a = (1, 1, 1)$, $\alpha = (\lambda, 1, 1)$, $\beta = (1, 1, 1)$ and hence

$$a\alpha = (1, 1, 1)(\lambda, 1, 1) = (1, 1, 1) = (1, 1, 1)(1, 1, 1) = a\beta. \tag{8}$$

However, for $b = (1, 1, i)$ we have

$$b\alpha = (1, 1, i)(\lambda, 1, 1) = (1, p_{i\lambda}, 1) \neq (1, 1, 1) = (1, 1, i)(1, 1, 1) = b\beta. \tag{9}$$

Thus, the equalities (8), (9) contradict (4), (5). ∎

An element s of a semigroup S is called *reducible* if there exist $a, b \in S$ with $s = ab$. Clearly, the set of all reducible elements $\mathrm{Red}(S)$ is an ideal of a semigroup S.

Lemma 3. Let S be a finite semigroup satisfying (4), (5). Then $\mathrm{Ker}(S)$ is the set of all reducible elements.

Proof. Since the kernel $\mathrm{Ker}(S)$ is simple, Theorem 2 gives $\mathrm{Ker}(S) = (G, \mathbf{P}, \Lambda, I)$ for some finite $G, \mathbf{P}, \Lambda, I$. Let $b \in S$. We have $(\lambda, g, i)b = (\lambda, g, i)(1, 1, i)b = (\lambda, g, i)r$, where $r = (1, 1, i)b \in \mathrm{Ker}(S)$. By (4), we obtain $ab = ar$ for any $a \in S$. Since $ar \in \mathrm{Ker}(S)$, so is ab. Thus, any product of elements belongs to $\mathrm{Ker}(S)$, hence $\mathrm{Red}(S) = \mathrm{Ker}(S)$. ∎

Theorem 4. If $\mathrm{Ker}(S) = \mathrm{Red}(S)$ for a finite homogroup S, then S satisfies (4), (5) or, equivalently, ΠS is $\mathcal{L}_{s-\mathrm{pred}}(\Pi S)$-equationally Noetherian.

Proof. Let us take a, b, α, β such that $a\alpha = a\beta$, and e be the identity of $\mathrm{Ker}(S)$. We have

$$\begin{aligned} a\alpha &= a\beta \mid \cdot e, \\ ea\alpha &= ea\beta, \\ (ea)\alpha &= (ea)\beta \mid \cdot (ea)^{-1} \text{ since } ea \text{ belongs to the group } \mathrm{Ker}(S), \\ e\alpha &= e\beta \mid e \text{ is a central element}, \\ \alpha e &= \beta e. \end{aligned}$$

We have (below we use $b\alpha, b\beta \in \mathrm{Ker}(S) = \mathrm{Red}(S)$):

$$b\alpha = (b\alpha)e = b(\alpha e) = b(\beta e) = (b\beta)e = b\beta.$$

Thus, the quasi-identity (4) holds for S. The proof for the quasi-identity (5) is similar. ■

One can directly check that for a rectangular band of groups $S = (G, \mathbf{P}, \Lambda, I)$ the analogue of Theorem 4 also holds.

Thus, there arises the following question.

Question. Suppose the kernel $\mathrm{Ker}(S)$ of a finite semigroup S satisfies the following conditions:

1) $\mathrm{Ker}(S) = \mathrm{Red}(S)$;
2) $\mathrm{Ker}(S)$ is a rectangular band of groups.

Does S satisfy the quasi-identities (4), (5)?

Example 1. The answer for the last question is negative. Let us consider a semigroup S with the following multiplication table:

$\cdot$	a	b	z_1	z_2	z_3	z_4
a	z_4	z_4	z_2	z_4	z_4	z_4
b	z_4	z_4	z_3	z_4	z_4	z_4
z_1	z_1	z_1	z_1	z_1	z_1	z_1
z_2	z_2	z_2	z_2	z_2	z_2	z_2
z_3	z_3	z_3	z_3	z_3	z_3	z_3
z_4	z_4	z_4	z_4	z_4	z_4	z_4

This Table defines an associative binary operation (we checked it by a computer).

One can directly compute that $\mathrm{Ker}(S) = \mathrm{Red}(S) = \{z_1, z_2, z_3, z_4\}$. Since the elements z_i are left zeros, we have $\mathrm{Ker}(S) = (G, \mathbf{P}, \Lambda, I)$, where $G = \{1\}$, $\mathbf{P} = (1, 1, 1, 1)$, $\Lambda = \{1, 2, 3, 4\}$, $I = \{1\}$. However, the quasi-identity (5) does not hold in S, since $az_1 = bz_1$, but $az_0 \neq bz_0$.

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