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Microformal Geometry and Homotopy Algebras

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Abstract—We extend the category of (super)manifolds and their smooth mappings by introducing a notion of microformal, or "thick," morphisms. They are formal canonical relations of a special form, constructed with the help of formal power expansions in cotangent directions. The result is a formal category so that its composition law is also specified by a formal power series. A microformal morphism acts on functions by an operation of pullback, which is in general a nonlinear transformation. More precisely, it is a formal mapping of formal manifolds of even functions (bosonic fields), which has the property that its derivative for every function is a ring homomorphism. This suggests an abstract notion of a "nonlinear algebra homomorphism" and the corresponding extension of the classical "algebraic–functional" duality. There is a parallel fermionic version. The obtained formalism provides a general construction of L_∞-morphisms for functions on homotopy Poisson (P_∞) or homotopy Schouten (S_∞) manifolds as pullbacks by Poisson microformal morphisms. We also show that the notion of the adjoint can be generalized to nonlinear operators as a microformal morphism. By applying this to L_∞-algebroids, we show that an L_∞-morphism of L_∞-algebroids induces an L_∞-morphism of the "homotopy Lie–Poisson" brackets for functions on the dual vector bundles. We apply this construction to higher Koszul brackets on differential forms and to triangular L_∞-bialgebroids. We also develop a quantum version (for the bosonic case), whose relation to the classical version is like that of the Schrödinger equation to the Hamilton–Jacobi equation. We show that the nonlinear pullbacks by microformal morphisms are the limits as $\hbar \to 0$ of certain "quantum pullbacks," which are defined as special form Fourier integral operators.

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INTRODUCTION

1. Generalization of pullbacks and homotopy brackets. Constructing L_∞-morphisms between L_∞-algebras is in general a difficult task; in some cases a particular example of an L_∞-morphism can represent a solution of a highly nontrivial problem such as Kontsevich's construction [23] of deformation quantization of Poisson manifolds.

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One of the results of this paper is a general method giving L_∞-morphisms for L_∞-algebras of functions. This is based on a certain extension, or "thickening," of the usual category of smooth manifolds or supermanifolds.

It is well known that the duality of the geometric ("functional") and algebraic viewpoints (see, e.g., [30]) plays an important role in many mathematical theories, sometimes as a heuristic principle, and sometimes in the form of precise statements and constructions, such as the Gelfand duality or Grothendieck's theory of schemes. By the geometric viewpoint, we mean a picture based on "spaces" (in one or another sense), and by the algebraic viewpoint, a picture based on algebras, treated as algebras of functions. Under this duality, maps of spaces correspond to algebra homomorphisms, so that to a map there corresponds the pullback of functions, $\varphi^*\colon g \mapsto \varphi^*(g) = g \circ \varphi$, which is a linear map preserving the multiplication, i.e., a homomorphism. In the present paper, we give constructions leading to a nonlinear generalization of such a duality.

We construct two formal categories extending the category of smooth (super)manifolds and smooth maps, with the same set of objects. Morphisms Φ in these formal categories, which we call *microformal* or *thick morphisms*, still act on smooth functions by a generalization of pullbacks. A key ingredient in the construction is an equation of the fixed point type, whose solution is obtained by iterations. Pullbacks by thick morphisms Φ^* are formal *nonlinear differential operators*, represented by perturbative series around ordinary pullbacks combined with additive shifts. Nonlinearity is the distinctive property of these new pullbacks. Similar equations and perturbative series arise for the composition law of thick morphisms (which is therefore formal).

Because of the nonlinearity, we have to distinguish functions that are of odd or even parity in the sense of the $\mathbb{Z}_2$-grading, as they have different commutativity properties. That is why there are two formal categories, so that morphisms in one of them, denoted $\mathcal{E}\mathfrak{T}\mathrm{hick}$, induce pullbacks of even functions ("bosonic fields"), while morphisms in the other, denoted $\mathcal{O}\mathfrak{T}\mathrm{hick}$, induce pullbacks of odd functions ("fermionic fields"). They are obtained by parallel constructions. Each of them contains the semidirect product category $\mathcal{S}\mathcal{M}\mathrm{an} \rtimes \mathbf{C}^\infty$ or $\mathcal{S}\mathcal{M}\mathrm{an} \rtimes \mathbf{\Pi C}^\infty$, respectively, as a closed subspace and can be regarded as its formal neighborhood. (Here $\mathcal{S}\mathcal{M}\mathrm{an}$ is the ordinary category of supermanifolds and $\mathbf{C}^\infty$ or $\mathbf{\Pi C}^\infty$ is the space of even or odd functions, on which smooth maps act by pullbacks.) There are embedding and retraction functors $\mathcal{S}\mathcal{M}\mathrm{an} \rtimes \mathbf{C}^\infty \rightleftarrows \mathcal{E}\mathfrak{T}\mathrm{hick}$ and $\mathcal{S}\mathcal{M}\mathrm{an} \rtimes \mathbf{\Pi C}^\infty \rightleftarrows \mathcal{O}\mathfrak{T}\mathrm{hick}$.

"Nonlinear pullbacks" were first introduced by us in [44] for the purpose of constructing L_∞-morphisms of homotopy Poisson algebras of functions (motivated by a problem for higher Koszul brackets [21]). Such an L_∞-morphism by definition should be a nonlinear map of functional supermanifolds, so it certainly cannot be a usual pullback. The idea of the construction of a "nonlinear pullback" was inspired by the cotangent philosophy of Kirill Mackenzie [27]. As we showed, these newly defined pullbacks with respect to thick morphisms indeed give the desired solution for homotopy Poisson brackets. Namely, if a thick morphism Φ is *Poisson*, which means that the master Hamiltonians or multivector fields specifying homotopy Schouten or Poisson structures are Φ-related (a condition expressed in coordinates by a Hamilton–Jacobi type equation), then the pullback map Φ^* is an L_∞-morphism of the algebras of functions.

2. Nonlinear algebraic–functional duality. As the pullback with respect to a thick morphism is a nonlinear transformation, it cannot be a ring homomorphism in the ordinary sense. It turns out, however, that its *derivative* at each function will be a ring homomorphism! Besides that, in spite of the nonlinearity, the pullbacks themselves exhibit some kind of duality similar to the classical case. For ordinary smooth maps, it is known that the pullbacks on functions determine a map completely; in particular, giving the pullbacks of coordinate functions is the same as specifying a map in coordinates. Similarly for a thick morphism, although it is not sufficient to know the images of individual coordinate functions, it is sufficient however to know the images of their linear combinations $\Phi^*[y^ic_i]$ with arbitrary parameters c_i. Another example of such a "nonlinear extension" from multiplicative generators is given by the pushforward of functions on the dual vector spaces or vector bundles by a nonlinear bundle map. We introduce it as the pullback with respect to the "adjoint operator," which, as we show, can be defined for a nonlinear map, but as a thick morphism rather than an ordinary map; as we show, on vectors or on sections of the original bundle this pushforward agrees with a given nonlinear mapping.

Algebraic properties of nonlinear pullbacks suggest the following abstract framework. For algebras A and B, define a *nonlinear homomorphism* as a smooth map of vector spaces $\alpha\colon A \to B$ such that the derivative $T\alpha(a)\colon A \to B$ at each $a \in A$ is an algebra homomorphism in the ordinary sense. (For superalgebras, one has to consider a map $\alpha\colon \mathbf{A} \to \mathbf{B}$ of the associated "linear supermanifolds" $\mathbf{A}$ and $\mathbf{B}$.) Similarly *formal homomorphisms* are defined. These notions should lead us to a nonlinear generalization of the algebraic–functional duality.

It can be asked whether every nonlinear (or formal) homomorphism between the algebras of smooth functions on (super)manifolds arises as the nonlinear pullback induced by some thick morphism. A positive answer would be a nonlinear counterpart of the well-known statement for ordinary homomorphisms and ordinary smooth maps.

3. Idea of construction. To construct the formal categories $\mathfrak{E}\mathfrak{T}$hick and $\mathfrak{O}\mathfrak{T}$hick and nonlinear pullbacks, we use very classical tools of mathematical physics such as canonical relations and their generating functions. To V. I. Arnold belongs a remark about the "unfortunately noninvariant" nature of generating functions [2, Sect. 47]. The positive interpretation of this fact is that generating functions possess a nontrivial transformation law under changes of coordinates. In our constructions, generating functions of a particular type appear as central geometric objects. A *thick morphism* between two supermanifolds is given by a generating function $S(x,q)$, which specifies a canonical relation between the corresponding cotangent bundles and is regarded as part of the structure. A generating function $S(x,q)$ is a function of positions on the source manifold and of momenta on the target manifold. The action on functions, $g(y) \mapsto f(x)$, is defined in terms of this generating function as

$$f(x) = g(y) + S(x,q) - y^i q_i,$$

where to eliminate the variables q and y one uses the coupled equations $q_i = \frac{\partial g}{\partial y^i}$ and $y^i = \frac{\partial S}{\partial q_i}$, solved by iterations. One can show that this formula generalizes the ordinary pullback (as the substitution into the argument). As the reader will see, we have to consider generating functions as formal power expansions in the momentum variables. This explains the adjective "microformal" in the alternative name for thick morphisms and the name *microformal geometry* for the whole theory.[1]

4. Plan of the paper. The exposition is organized as follows.

In Section 1, we introduce the *microformal categories* $\mathfrak{E}\mathfrak{T}$hick and $\mathfrak{O}\mathfrak{T}$hick, and develop the functorial properties of thick morphisms (the construction of pullback).

In Section 2, we define the adjoint for a nonlinear morphism of vector spaces or vector bundles as a thick morphism of the dual bundles, with properties similar to those of the ordinary adjoints. The construction uses the canonical diffeomorphism $T^*E \cong T^*E^*$ of Mackenzie and Xu [28] and its odd analog $\Pi T^*E \cong \Pi T^*(\Pi E^*)$ introduced in [36]. Using them, we prove in Section 3 that an L_∞-morphism of L_∞-algebroids induces L_∞-morphisms of the homotopy Lie–Poisson brackets on the dual vector bundles and Lie–Schouten brackets on the antidual vector bundles. We then apply this result to the theory of higher Koszul brackets and to triangular L_∞-bialgebroids.

In Section 4, we show that, in the bosonic case, the microformal category and nonlinear pullbacks are the classical limit (for $\hbar \to 0$) of a *quantum microformal category*, which is dual to a category

[1] The prefix "micro-" has an established usage, e.g., in microlocal analysis (local in the cotangent or jet directions) and Milnor's microbundles. It is also used in "symplectic microgeometry" [4–6].

whose morphisms are a particular type of Fourier integral operators perceived as "quantum pullbacks." Each such operator is specified by a "quantum generating function." Quantum pullbacks act on oscillatory wave functions, which are linear combinations of oscillatory exponentials with coefficients in formal power series in $\hbar$. Calculating the integrals by the stationary phase method yields formulas for "classical" thick morphisms. In hindsight, one may see this as a justification of the "classical" formulas. Finally, in Section 5, we show how the applications of thick morphisms to homotopy bracket structures can be lifted to the "quantum" level.

Since the quantum version of our constructions relies on the stationary phase method, we included an appendix containing the necessary statements in the form adapted for our purposes.

One clarifying remark is in order, that two different types of formal power expansions arise here. One expansion is present already in the classical theory (generating functions themselves, pullback, composition law). It can be compared with the "expansion in the coupling constant" in field theory. Another is the expansion in $\hbar$ and gives "quantum corrections."

We also wish to point out a relation between this "microformal geometry" and the "symplectic microgeometry" of A. Cattaneo, B. Dherin and A. Weinstein. In a remarkable series of papers [4–6] (see also [7, 51]), they systematically developed a "micro" analog of symplectic geometry with "symplectic microfolds" defined as germs of symplectic manifolds at Lagrangian submanifolds and with germs of canonical relations as morphisms. The microsymplectic category so obtained was intended to cure the problem of partially defined multiplication in Weinstein's symplectic "category in quotes" [45–50]. Our formal categories $\mathfrak{E}\mathfrak{Thick}$ and $\mathfrak{O}\mathfrak{Thick}$ are close to this microsymplectic category. The key difference is that in our case, (formal) canonical relations between the cotangent bundles play the role of morphisms between the bases—not between the bundles themselves—and they are introduced in order to obtain an action on smooth functions on the bases, which is our central concept of nonlinear pullback.[2]

5. Terminology and notations. For simplicity, we often use "manifolds" for "supermanifolds" and generally suppress the prefix "super-" unless we wish to emphasize that we consider the super case. Also, to simplify the speech, we as a rule suppress the prefix "pseudo-" and speak about differential forms and multivector fields, on a supermanifold, when, strictly speaking, pseudodifferential forms and pseudomultivector fields are discussed (i.e., by definition, arbitrary smooth functions on the bundles ΠTM and ΠT^*M, respectively). In the notation and terminology we generally follow [36–40]. The parity ($\mathbb{Z}_2$-grading) of an object is denoted by a tilde over its symbol. Tensor indices carry the parities of the corresponding coordinates. The symbol Π stands for the parity reversion functor on vector spaces, modules or vector bundles. For a substantial part of our constructions, the supergeometric context is inessential. Consideration of supermanifolds is necessary for applications to homotopy structures. For applications, one may also need *graded manifolds*, which are supermanifolds that besides the $\mathbb{Z}_2$-grading, or *parity*, possess an independent $\mathbb{Z}$-grading, or *weight* (see [36] as well as [38–40]). Our constructions can be extended to the graded case without difficulty.

Throughout the paper we denote local coordinates on a manifold M by x^a and the canonically conjugate momenta by p_a. The canonical symplectic form on T^*M is

$$\omega = dp_a\, dx^a = d(p_a\, dx^a).$$

Note that the *Liouville* 1*-form* $\theta = p_a\, dx^a$ is defined invariantly. When we need several manifolds, we introduce different letters for local coordinates on each of them, as well as for the corresponding conjugate momenta.

[2]This action on functions on Lagrangian submanifolds in the ambient symplectic manifolds brings to mind the spinor representation in its various versions; it is curious to clarify whether this is more than a superficial resemblance.

1. "EVEN" AND "ODD" MICROFORMAL CATEGORIES. MAIN PROPERTIES

Consider supermanifolds M_1 and M_2 with local coordinates x^a and y^i, and the corresponding conjugate momenta p_a and q_i (coordinates on the cotangent spaces). Let $T^*M_2 \times (-T^*M_1)$ denote the product $T^*M_2 \times T^*M_1$ equipped with the symplectic form[3]

$$\omega = \omega_2 - \omega_1 = d(q_i\, dy^i - p_a\, dx^a).$$

Definition 1. A *thick morphism* (or *microformal morphism*) $\Phi\colon M_1 \Rrightarrow M_2$ is defined as a formal canonical relation (which we denote by the same letter) $\Phi \subset T^*M_2 \times (-T^*M_1)$, together with an even function $S = S(x,q)$ defined in each local coordinate system and depending on (having as arguments) position variables on the source manifold and momentum variables on the target manifold, such that

$$\Phi = \left\{ (y^i, q_i; x^a, p_a) \;\Big|\; y^i = (-1)^{\tilde{i}} \frac{\partial S}{\partial q_i}(x,q),\ p_a = \frac{\partial S}{\partial x^a}(x,q) \right\}. \tag{1.1}$$

We call the function $S = S(x,q)$ the *generating function* of a thick morphism. It is considered part of the structure.

We shall elaborate this definition below, but first give an example.

Example 1. Consider a smooth map $\varphi\colon M_1 \to M_2$. In local coordinates, it is given by $y^i = \varphi^i(x)$. Set

$$S(x,q) = \varphi^i(x) q_i. \tag{1.2}$$

This function gives a canonical relation $R_\varphi \subset T^*M_2 \times (-T^*M_1)$ specified by the equations (as one immediately sees)

$$y^i = \varphi^i(x), \qquad p_a = \frac{\partial \varphi^i}{\partial x^a}(x) q_i.$$

The relation R_φ is the canonical lifting of a map φ to the cotangent bundles. In a coordinate-free language,

$$R_\varphi = \mathrm{graph}(\overline{\varphi}) \circ (\mathrm{graph}(T^*\varphi))^{\mathrm{op}},$$

where $\overline{\varphi}\colon \varphi^*(T^*M_2) \to T^*M_2$ is the vector bundle morphism which is identity on the fibers and covers a map of the bases $\varphi\colon M_1 \to M_2$, and $T^*\varphi\colon \varphi^*(T^*M_2) \to T^*M_1$ is the dual to the tangent map $T\varphi\colon TM_1 \to \varphi^*(TM_2)$. Here $(\mathrm{graph}(T^*\varphi))^{\mathrm{op}}$ is the opposite relation for $\mathrm{graph}(T^*\varphi) \subset T^*M_1 \times \varphi^*(T^*M_2)$, so the composition R_φ is indeed a submanifold in $T^*M_2 \times T^*M_1$.

It follows that we can identify ordinary smooth maps $\varphi\colon M_1 \to M_2$ with a subclass of thick morphisms $M_1 \Rrightarrow M_2$ specified by generating functions $S(x,q)$ of the form (1.2), i.e., linear in momenta.

Consider now the general case. Recall that a canonical relation (or correspondence) Φ between symplectic manifolds N_1 and N_2 (in our case these are T^*M_1 and T^*M_2) is a Lagrangian submanifold in the product $N_2 \times (-N_1)$ taken with the form $\omega = \omega_2 - \omega_1$. Such relations are customarily perceived as partial multivalued mappings $N_1 \dashrightarrow N_2$ (direction of the arrow being a matter of convention) that generalize symplectomorphisms. However, this is not the intuition that we shall follow. For us this relation (or correspondence) Φ is an analog of a map between the manifolds M_1 and M_2 themselves (and not between their cotangent bundles). For our purposes we consider not arbitrary canonical relations but only of a particular kind, those that are specified by generating

[3] We have changed notations in comparison with [44], where $T^*M_1 \times (-T^*M_2)$ was used. The order $T^*M_2 \times (-T^*M_1)$ is more traditional in symplectic geometry. Note that it is also convenient to regard the graphs of maps $f\colon X \to Y$ as subspaces of $Y \times X$, not $X \times Y$.

functions of the type $S(x,q)$. (In particular, unlike relations in general, the direction from M_1 to M_2 in our constructions is unambiguous and not a matter of convention.)

To understand the role of the generating function S in Definition 1, recall that for an arbitrary Lagrangian submanifold $\Phi\subset T^*M_2\times(-T^*M_1)$ the 1-form $q_i\,dy^i-p_a\,dx^a$ is closed, hence locally exact; i.e., there is a function F on Φ defined independently of a choice of coordinates (but possibly only locally and up to a constant), such that

$$q_i\,dy^i-p_a\,dx^a=dF. \tag{1.3}$$

In Definition 1 it is assumed that the variables x^a and q_i yield a system of local coordinates on the submanifold Φ. (This follows the case of an ordinary map.) The equations specifying Φ mean that $p_a\,dx^a+(-1)^{\tilde\imath}y^i\,dq_i=dS$, which is equivalent to

$$q_i\,dy^i-p_a\,dx^a=d(y^iq_i-S). \tag{1.4}$$

The left-hand side of (1.4) is invariant, but the explicit appearance of the variables y^i and q_i on the right-hand side makes the function $S(x,q)$ a coordinate-dependent object (unlike F in (1.3)). We shall give below the precise transformation law for S. The functions S and F are related by a Legendre transform type formula,

$$F=y^iq_i-S. \tag{1.5}$$

(It is an actual Legendre transform if F can be regarded as a function of independent variables x^a,y^i, which may not necessarily hold in general.) The relation Φ defines only the differentials dF or dS. We assume that constants of integration are chosen, so that we can speak unambiguously about the functions F or S, and that F can be defined globally.

What is a coordinate-free characterization of the considered type of Lagrangian submanifolds? The condition that x^a be independent on Φ is equivalent to the submanifold Φ projecting on M_1 without degeneration (with full rank). In contrast with that, the second condition, that q_i be independent on Φ, is equivalent to Φ "projecting without degeneration on the fibers of T^*M_2," but this seems to not have a well-defined meaning without a choice of a local trivialization. Consider, however, the differentials dq_i. We have $q_i=\frac{\partial y^{i'}}{\partial y^i}\,q_{i'}$, so obtain

$$dq_i=d\left(\frac{\partial y^{i'}}{\partial y^i}\right)q_{i'}+(1)^{\tilde\imath+\tilde\imath'}\,\frac{\partial y^{i'}}{\partial y^i}\,dq_{i'}.$$

We see that when $q_{i'}$ are small (i.e., we are near the zero section of T^*M_2), the linear independence of dq_i on Φ implies the linear independence of $dq_{i'}$, and vice versa. Therefore, we conclude that the condition that the variables q_i be independent on Φ (or "Φ project without degeneration on the fibers of T^*M_2") has an invariant meaning on a small neighborhood of the zero section of T^*M_2. In particular, it makes sense on the formal neighborhood of M_2 in T^*M_2. Therefore, we define Φ as a *formal* canonical relation, i.e., a Lagrangian submanifold of the formal neighborhood[4] of $M_2\times T^*M_1$ in $T^*M_2\times(-T^*M_1)$.

Hence we consider the generating function $S(x,q)$ of a thick morphism $\Phi\colon M_1 \twoheadrightarrow M_2$ as a formal power series

$$S(x,q)=S^0(x)+S^i(x)q_i+\frac{1}{2}\,S^{ij}(x)q_jq_i+\frac{1}{3!}\,S^{ijk}(x)q_kq_jq_i+\dots \tag{1.6}$$

[4]Replacing a formal submanifold by a germ would give a "symplectic micromorphism" between "symplectic microfolds" represented by the pairs (T^*M_1,M_1) and (T^*M_2,M_2), a notion introduced by Cattaneo, Dherin and Weinstein. Note that our thick morphisms are morphisms between M_1 and M_2, while symplectic micromorphisms are morphisms between objects of double dimensions.

in the momentum variables q_i. In the sequel we frequently suppress the adjective "formal" for various objects that we consider (functions, submanifolds, etc.). As we shall see, it makes sense to group the terms in this expansion as

$$S(x,q) = S^0(x) + S^i(x)q_i + S^+(x,q), \tag{1.7}$$

where $S^+(x,q)$ contains all terms of order 2 and higher in q_i.

To conclude elaborating our definition, we state the following transformation law for the generating functions S. For logical simplicity we may regard it as part of the definition, but it can be deduced from equations (1.3)–(1.5) together with the invariance condition for a submanifold Φ.

Transformation law (for generating functions)**.** A *generating function* S of a thick morphism $\Phi\colon M_1 \Rrightarrow M_2$ as a geometric object on $M_1 \times M_2$ transforms by

$$S'(x',q') = S(x,q) - y^i q_i + y^{i'} q_{i'} \tag{1.8}$$

under an invertible change of local coordinates $x^a = x^a(x')$, $y^i = y^i(y')$. Here $S(x,q)$ is the expression for S in "old" coordinates and $S'(x',q')$ is the expression for S in "new" coordinates. The variables x^a and $y^{i'}$ on the right-hand side of (1.8) are given simply by the substitutions $x^a = x^a(x')$ and $y^{i'} = y^{i'}(y)$ (where, as usual, $y^{i'} = y^{i'}(y)$ is the inverse change of coordinates), while q_i and y^i are determined from the coupled equations

$$q_i = \frac{\partial y^{i'}}{\partial y^i}(y)\, q_{i'}, \qquad y^i = (-1)^{\tilde{\imath}}\, \frac{\partial S}{\partial q_i}(x,q). \tag{1.9}$$

Proposition 1. *The transformation law* (1.8) *satisfies the cocycle condition* (*hence, in particular, the set of generating functions S is nonempty*)*. A generating function S with local representations $S(x,q)$ and the transformation law* (1.8) *specifies a well-defined formal canonical relation $\Phi \subset T^*M_2 \times (-T^*M_1)$.*

Proof. The cocycle condition immediately follows because the transformation law (1.8) has a "coboundary" form. Equation (1.8) also means that the functions $y^i q_i - S(x,q)$ glue into one global function. To check the second statement, we need to show that if (1.4) holds for S, x^a, p_a, y^i and q_i and S' is related to S by the given transformation law, then the same relation

$$q_{i'}\, dy^{i'} - p_{a'}\, dx^{a'} = d(y^{i'} q_{i'} - S') \tag{1.10}$$

holds for the "new" variables S', $x^{a'}$, $p_{a'}$, $y^{i'}$ and $q_{i'}$ (assuming the standard transformation laws for the positions and momenta). But the left-hand side of (1.10) equals $q_i\, dy^i - p_a\, dx^a$ by the invariance of the Liouville forms, and $y^{i'} q_{i'} - S'$ on the right-hand side equals $y^i q_i - S$ by (1.8). Hence, (1.4) and (1.10) are equivalent. $\square$

Example 2. Consider a generating function S that in one coordinate system has the form

$$S(x,q) = S^0(x) + \varphi^i(x) q_i \tag{1.11}$$

(we write φ^i instead of S^i for convenience, as will become clear shortly). Explore the action of the transformation law on S. We have

$$S'(x',q') = S(x,q) - y^i q_i + y^{i'} q_{i'} = S^0(x) + \varphi^i(x) q_i - y^i q_i + y^{i'} q_{i'},$$

where we should substitute $x = x(x')$ and $y' = y'(y)$; and for y and q' we need to solve equations (1.9). But in our case, they decouple and for y simply give

$$y^i = (-1)^{\tilde{\imath}}\, \frac{\partial S}{\partial q_i}(x,q) = \varphi^i(x).$$

Hence the terms $\varphi^i(x)q_i - y^iq_i$ in S' cancel and we obtain (taking into account the substitutions $y'=y'(y)$, $y=\varphi(x)$ and $x=x(x')$)

$$S'(x',q') = S^0(x) + y^{i'}q_{i'} = S^0(x(x')) + y^{i'}(\varphi(x(x')))q_{i'}.$$

In other words, in new coordinates S has the same form

$$S'(x',q') = S'^0(x') + \varphi^{i'}q_{i'},$$

where

$$S'^0(x') = S^0(x(x')) \qquad \text{and} \qquad \varphi^{i'} = y^{i'}(\varphi(x(x'))).$$

These are precisely the transformation laws for coordinate representations of a scalar function on M_1 and a map $\varphi\colon M_1\to M_2$.

We conclude that thick morphisms $\Phi\colon M_1 \Rrightarrow M_2$ with generating functions S of the form (1.11), of degree ≤ 1 in momenta, invariantly correspond to pairs (φ, S^0) where $\varphi\colon M_1\to M_2$ is a smooth map and $S^0\in C^\infty(M_1)$ is an (even) smooth function on the source manifold. (We shall see later that such pairs are morphisms in a semidirect product category.)

Example 3. Consider now the general case where the generating function of a thick morphism $\Phi\colon M_1 \Rrightarrow M_2$ has the form (1.7). We rewrite it as

$$S(x,q) = S^0(x) + \varphi^i(x)q_i + S^+(x,q), \tag{1.12}$$

having in mind the previous example. Let us analyze how the particular terms in (1.12) transform. The transformation law gives

$$S'(x',q') = S(x,q) - y^iq_i + y^{i'}q_{i'} = S^0(x) + \varphi^i(x)q_i + S^+(x,q) - y^iq_i + y^{i'}q_{i'},$$

where as before we have to substitute $x=x(x')$ and $y'=y'(y)$; and y and q are obtained by solving equations (1.9). But now the equation for determining y takes the form

$$y^i = \varphi^i(x) + (-1)^{\tilde{\imath}}\frac{\partial S^+}{\partial q_i}\biggl(x, \frac{\partial y'}{\partial y}(y)q'\biggr);$$

note that the second term is of order ≥ 1 in q'. This gives a unique solution as a power series in q', of the form

$$y^i = \varphi^i(x) + y^{+i}(x,q')$$

(the second term is of order ≥ 1 in q'). Hence

$$q_i = \frac{\partial y^{i'}}{\partial y^i}\bigl(\varphi(x) + y^+(x,q')\bigr)q_{i'},$$

and for S' we arrive at

$$\begin{aligned}S'(x',q') &= S^0(x) + (\varphi^i(x) - y^i)q_i + S^+(x,q) + y^{i'}q_{i'}\\ &= S^0(x) - y^{+i}(x,q')\,q_i + S^+(x,q) + y^{i'}\bigl(\varphi(x)+y^+(x,q')\bigr)q_{i'}\\ &= S^0(x) + y^{i'}\bigl(\varphi(x)+y^+(x,q')\bigr)q_{i'} - y^{+i}(x,q')\,\frac{\partial y^{i'}}{\partial y^i}\bigl(\varphi(x)+y^+(x,q')\bigr)q_{i'}\\ &\quad + S^+\biggl(x, \frac{\partial y'}{\partial y}\bigl(\varphi(x)+y^+(x,q')\bigr)q'\biggr),\end{aligned}$$

where we need to substitute finally $x=x(x')$. In particular, we obtain

$$S'(x',q') \equiv S^0(x(x')) + y^{i'}(\varphi(x(x')))q_{i'} \mod \langle q'\rangle^2.$$

Hence

$$S'(x',q') = S'^0(x') + \varphi^{i'}(x')q_{i'} + S'^+(x',q'),$$

where

$$S'^0(x') = S^0(x(x')) \qquad \text{and} \qquad \varphi^{i'} = y^{i'}(\varphi(x(x'))).$$

This means that the first two terms in the expansion (1.7) or (1.12) represent, respectively, a scalar function on M_1 and a map $\varphi\colon M_1 \to M_2$. At the same time, the transformation law for the term S^+ includes higher derivatives of changes of coordinates on M_2 calculated at the points $\varphi(x)$.

From Examples 2 and 3, we see that pairs (φ, S^0) correspond to thick morphisms $M_1 \Rrightarrow M_2$ of a special type and, conversely, an arbitrary thick morphism $\Phi\colon M_1 \Rrightarrow M_2$ canonically defines such a pair. So we have an "inclusion–retraction" setting. We shall come back to that.

Our next task is to define the action of thick morphisms on functions.

Consider the algebras of smooth functions $C^\infty(M)$. For each supermanifold M, the algebra $C^\infty(M)$ is a commutative $\mathbb{Z}_2$-graded algebra. We shall regard smooth functions of particular parity on M as points of an infinite-dimensional supermanifold. (The word "smooth" will often be omitted in the sequel.) We have the *supermanifold of all even functions* on M, which we denote by $\mathbf{C}^\infty(M)$, and the *supermanifold of all odd functions* on M, which we denote by $\mathbf{\Pi C}^\infty(M)$. We use boldface to distinguish vector supermanifolds from the $\mathbb{Z}_2$-graded linear spaces corresponding to them. (A physicist would say that the points of $\mathbf{C}^\infty(M)$ are "bosonic fields" and the points of $\mathbf{\Pi C}^\infty(M)$ are "fermionic fields" on M.)

Definition 2 [44]**.** Let $\Phi\colon M_1 \Rrightarrow M_2$ be a thick morphism with a generating function S. The *pullback* Φ^* is a formal mapping of functional supermanifolds of even functions, $g \mapsto \Phi^*[g]$,

$$\Phi^*\colon\ \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1), \tag{1.13}$$

defined by

$$\Phi^*[g](x) = g(y) + S(x,q) - y^i q_i, \tag{1.14}$$

where q_i and y^i are determined from the equations

$$q_i = \frac{\partial g}{\partial y^i}(y) \qquad \text{and} \qquad y^i = (-1)^{\tilde{i}}\frac{\partial S}{\partial q_i}(x,q). \tag{1.15}$$

Here $g \in \mathbf{C}^\infty(M_2)$ is an even function on M_2 and $\Phi^*[g]$ is its image in $\mathbf{C}^\infty(M_1)$.

Remark 1. We showed in [44] that the pullback Φ^* does not depend on a choice of coordinates. This is guaranteed by the transformation law of the generating function S.

Example 4. Consider a thick morphism $\Phi\colon M_1 \Rrightarrow M_2$ defined by a pair (φ, S^0). We have

$$S(x,q) = S^0(x) + \varphi^i(x)q_i.$$

From the second equation in (1.15), we obtain $y^i = \varphi^i(x)$, so

$$\Phi^*[g](x) = g(y) + S(x,q) - y^i q_i = g(y) + S^0(x) + \varphi^i(x)q_i - y^i q_i = g(\varphi(x)) + S^0(x).$$

Hence Φ^* in this case is an affine transformation,

$$\Phi^*[g] = S^0 + \varphi^*(g), \tag{1.16}$$

the combination of the ordinary pullback by a map $\varphi\colon M_1 \to M_2$ and the shift by a function $S^0 \in \mathbf{C}^\infty(M_1)$. (In particular, formula (1.14) gives the usual pullback when a thick morphism is an ordinary smooth map.)

Let us see how the construction of Φ^* works in general.

Substituting the first equation in (1.15) into the second gives the equation for y^i,

$$y^i = (-1)^{\tilde{\imath}}\,\frac{\partial S}{\partial q_i}\biggl(x,\frac{\partial g}{\partial y}(y)\biggr), \tag{1.17}$$

which can be solved by iterations. If we use (1.12), the equation takes the form

$$y^i = \varphi^i(x) + (-1)^{\tilde{\imath}}\,\frac{\partial S^+}{\partial q_i}\biggl(x,\frac{\partial g}{\partial y}(y)\biggr), \tag{1.18}$$

where the second term is of order ≥ 1 in $\frac{\partial g}{\partial y}$. There is a unique solution for y as a "functional" power series in g. More precisely, this is a formal power series in the first and higher derivatives of g evaluated at $y=\varphi(x)$ and starting from $y=\varphi(x)$ as the zero-order term. This gives a "perturbed" map $\varphi_g\colon M_1\to M_2$ depending on $g\in \mathbf{C}^\infty(M_2)$ as a series

$$\varphi_g = \varphi + \varphi_{(1)} + \varphi_{(2)} + \ldots, \tag{1.19}$$

where $\varphi\colon M_1\to M_2$ is defined by the thick morphism Φ and does not depend on g, while the next terms $\varphi_{(k)}$ give "higher corrections" to φ (linear, quadratic, etc., in the function g). Using φ_g, one can express the pullback $\Phi^*[g]$ as

$$\Phi^*[g](x) = g(\varphi_g(x)) + S\biggl(x,\frac{\partial g}{\partial y}(\varphi_g(x))\biggr) - \varphi_g^i(x)\,\frac{\partial g}{\partial y^i}(\varphi_g(x)), \tag{1.20}$$

which demonstrates the nonlinear dependence on g. In terms of (1.12), after simplification we obtain

$$\begin{aligned}\Phi^*[g](x) &= S^0(x) + g\bigl(\varphi(x)+\varphi_{(1)}(x)+\ldots\bigr)\\ &\quad - \bigl(\varphi^i_{(1)}(x)+\ldots\bigr)\,\frac{\partial g}{\partial y^i}\bigl(\varphi(x)+\varphi_{(1)}(x)+\ldots\bigr) + S^+\biggl(x,\frac{\partial g}{\partial y}\bigl(\varphi(x)+\varphi_{(1)}(x)+\ldots\bigr)\biggr).\end{aligned} \tag{1.21}$$

Example 5. Calculate $\Phi^*[g]$ to the second order in g. From (1.21), we immediately see that the terms of order ≤ 1 are precisely

$$S^0(x) + g(\varphi(x)).$$

For the quadratic correction, there are inputs from the three last summands in (1.21), but two of them cancel:

$$\varphi^i_{(1)}(x)\frac{\partial g}{\partial y^i}(\varphi(x)) - \varphi^i_{(1)}(x)\frac{\partial g}{\partial y^i}(\varphi(x)) + S^+_{(2)}\biggl(x,\frac{\partial g}{\partial y}(\varphi(x))\biggr) = S^+_{(2)}\biggl(x,\frac{\partial g}{\partial y}(\varphi(x))\biggr).$$

Here $S^+_{(2)}(x,q)=\frac{1}{2}S^{ij}(x)q_jq_i$ is the quadratic term in the expansion of S. Altogether,

$$\Phi^*[g](x) = S^0(x) + g(\varphi(x)) + \frac{1}{2}\,S^{ij}(x)\,\partial_i g(\varphi(x))\,\partial_j g(\varphi(x)) + \ldots. \tag{1.22}$$

This is the general pattern: the pullback $\Phi^*\colon \mathbf{C}^\infty(M_2)\to\mathbf{C}^\infty(M_1)$ with respect to a thick morphism is a formal nonlinear differential operator, so that the terms of order k in g of the expansion of $\Phi^*[g]$ are homogeneous polynomials of degree k in the derivatives of g of orders $\leq k-1$ evaluated at $y=\varphi(x)$, with the zero- and first-order terms being the combination of the shift and ordinary pullback: $S^0+\varphi^*(g)$. We again see the different roles of the three summands in the expansion (1.7), (1.12).

Remark 2. Pullbacks with respect to thick morphisms can be applied to functions defined on an open domain $U \subset M_2$. The image will be in $\mathbf{C}^\infty(\varphi^{-1}(U))$, where $\varphi\colon M_1 \to M_2$ is the underlying ordinary map.

Example 6. If we apply Φ^* to the function $g = y^i c_i$, where y^i are local coordinates on M_2 and c_i are some auxiliary variables, then we obtain $q_i = c_i$ from the first equation in (1.15) and

$$\Phi^*[y^i c_i] = y^i c_i - y^i q_i + S(x, q) = S(x, c).$$

In this way we recover the generating function $S = S(x, q)$.

A thick morphism Φ is therefore determined by the action of Φ^* on linear combinations of coordinate functions. Hence, although the pullback Φ^* is a nonlinear mapping, it still respects some algebraic properties such as the role of local coordinates as "free generators."[5]

In [44], we proved the following statement that points to another aspect of algebraic properties of pullbacks Φ^*.

Theorem 1 [44]**.** *For every function $g \in \mathbf{C}^\infty(M_2)$, the tangent map*

$$T\Phi^*[g]\colon\ C^\infty(M_2) \to C^\infty(M_1)$$

for the pullback $\Phi^*\colon \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$ by a thick morphism $\Phi\colon M_1 \Rrightarrow M_2$ is the ordinary pullback φ_g^* by the map

$$\varphi_g\colon\ M_1 \to M_2$$

that corresponds to the function g. $\square$

(Note that tangent spaces to $\mathbf{C}^\infty(M)$ can be identified with $C^\infty(M)$.)

We see that though the pullback with respect to a thick morphism as a mapping between the vector supermanifolds corresponding to the algebras of smooth functions is in general a nonlinear (and indeed formal) mapping, and as such cannot be an algebra homomorphism in the usual sense, it possesses the remarkable property that its derivative (= tangent map or linearization) at every point is an algebra homomorphism. It is tempting to give the following definition.

Definition 3. Let A and B be (super)algebras and $\mathbf{A}$ and $\mathbf{B}$ denote the corresponding vector supermanifolds. A map (formal map) $\alpha\colon \mathbf{A} \to \mathbf{B}$ is a *nonlinear algebra homomorphism* (respectively, a *formal nonlinear algebra homomorphism*) if its derivative $T\alpha(\mathbf{a})\colon A \to B$ is an algebra homomorphism for every $\mathbf{a} \in \mathbf{A}$.

(The distinction between A and $\mathbf{A}$, as well as B and $\mathbf{B}$, is important only in the super case.)

Pullbacks with respect to thick morphisms are formal nonlinear algebra homomorphisms. (In the abstract case, it is unclear whether formal or informal version of the notion is more important.) Following the known statement for ordinary algebra homomorphisms, we are tempted to suggest a conjecture.

Conjecture 1. *For smooth* (*super*)*manifolds M_1 and M_2* (with the usual assumptions leading to paracompactness), *every formal nonlinear algebra homomorphism*

$$\alpha\colon\ \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$$

is the pullback, $\alpha = \Phi^*$, with respect to some thick morphism

$$\Phi\colon\ M_1 \Rrightarrow M_2.$$

(So far we do not know whether this is true or not.)

[5] The algebra of smooth functions on a coordinate (super)domain is not of course a free algebra in the standard algebraic sense with respect to arbitrary homomorphisms (which would be the polynomial algebra), but it behaves as a free algebra with respect to the homomorphisms induced by smooth maps, which are defined by the images of the coordinate functions not subject to any restrictions.

Now we wish to establish categorical properties of thick morphisms.

Consider thick morphisms $\Phi_{21}\colon M_1 \to M_2$ and $\Phi_{31}\colon M_2 \to M_3$ with generating functions $S_{21} = S_{21}(x,q)$ and $S_{32} = S_{32}(y,r)$, respectively. Here z^μ are local coordinates on M_3 and by r_μ we denote the corresponding conjugate momenta.

Theorem 2. *The composition $\Phi_{32}\circ\Phi_{21}$ is well defined as a thick morphism*

$$\Phi_{31}\colon\ M_1 \to M_3$$

with the generating function $S_{31} = S_{31}(x,r)$, where

$$S_{31}(x,r) = S_{32}(y,r) + S_{21}(x,q) - y^i q_i \tag{1.23}$$

and y^i and q_i are expressed through (x^a, r_μ) from the system

$$q_i = \frac{\partial S_{32}}{\partial y^i}\,(y,r) \qquad \textit{and} \qquad y^i = (-1)^{\tilde{\imath}}\,\frac{\partial S_{21}}{\partial q_i}\,(x,q), \tag{1.24}$$

which has a unique solution as a power series in r_μ and a functional power series in S_{32}.

Proof. To find the composition of Φ_{32} and Φ_{21} as relations $\Phi_{32}\subset T^*M_3\times T^*M_2$ and $\Phi_{21}\subset T^*M_2\times T^*M_1$, we need to consider all pairs $(z,r;x,p)\in T^*M_3\times T^*M_1$ for which there exist $(y,q)\in T^*M_2$ such that $(z,r;y,q)\in\Phi_{32}\subset T^*M_3\times T^*M_2$ and $(y,q;x,p)\in\Phi_{21}\subset T^*M_2\times T^*M_1$. By the definition of Φ_{21}, we should have

$$y^i = (-1)^{\tilde{\imath}}\,\frac{\partial S_{21}}{\partial q_i}\,(x,q),$$

where x^a and q_i are free variables, and by the definition of Φ_{32}, we should have

$$q_i = \frac{\partial S_{32}}{\partial y^i}\,(y,r),$$

where now y^i and r_μ are free variables. Therefore, we arrive at system (1.24), where y^i and q_i are to be determined and the variables x^a and r_μ are free. Substituting the first equation in (1.24) into the second, we obtain for determining y the equation

$$y^i = (-1)^{\tilde{\imath}}\,\frac{\partial S_{21}}{\partial q_i}\left(x, \frac{\partial S_{32}}{\partial y}\,(y,r)\right),$$

which has a unique solution $y^i = y^i(x,r)$ by iterations, similarly to the construction of pullback. Here the “parameter of smallness” is S_{32}, more precisely, its derivative in y^i in the lowest order in r_μ. The solution for y^i can be substituted back into the first equation in (1.24) to obtain an expression $q_i = q_i(x,r)$. It remains to show that this composition of relations is indeed specified by the generating function given by (1.23). We have

$$q_i\,dy^i - p_a\,dx^a = d(y^i q_i - S_{21}) \qquad \text{and} \qquad r_\mu dz^\mu - q_i\,dy^i = d(z^\mu r_\mu - S_{32}).$$

We obtain

$$r_\mu\,dz^\mu - p_a\,dx^a = d\big(z^\mu r_\mu - S_{32} + y^i q_i - S_{21}\big).$$

Therefore, $S_{31} = S_{32} - y^i q_i + S_{21}$, as claimed. $\square$

Theorem 3. *The composition of thick morphisms is associative.*

Proof. Consider the diagram

$$M_1 \xrightarrow{\Phi_{21}} M_2 \xrightarrow{\Phi_{32}} M_3 \xrightarrow{\Phi_{43}} M_4.$$

Let $\Phi_{42}=\Phi_{43}\circ\Phi_{32}$ and $\Phi_{31}=\Phi_{32}\circ\Phi_{21}$. We need to check that $\Phi_{43}\circ\Phi_{31}=\Phi_{42}\circ\Phi_{21}$. Consider the generating functions. For the left-hand side, we obtain $S_{43}+S_{31}-z^\mu r_\mu=S_{43}+S_{32}+S_{21}-y^iq_i-z^\mu r_\mu$. For the right-hand side, we obtain $S_{42}+S_{21}-y^iq_i=S_{43}+S_{32}-z^\mu r_\mu+S_{21}-y^iq_i$, and the associativity follows. $\square$

Remark 3. Since there is an identity thick morphism for each supermanifold M, given by the generating function $S=x^aq_a$, we conclude that thick morphisms form a *formal category*, which we denote by $\mathcal{E}\mathfrak{T}\mathrm{hick}$ (with the same set of objects as the usual category of supermanifolds). "Formality" of the category means that the composition law is given by a power series. Formality enters our constructions in two related but different ways: as *micro*formality, i.e., power expansions in the cotangent directions, and as formal "functional" expansions in the formulas for pullback and for the generating function of composition.

Example 7. Let us compute the composition of thick morphisms in the lowest order. Suppose Φ_{21} and Φ_{32} are given by generating functions

$$S_{21}(x,q)=f_{21}(x)+\varphi_{21}^i(x)q_i+\dots,\qquad S_{32}(y,r)=f_{32}(y)+\varphi_{32}^\mu(y)r_\mu+\dots. \tag{1.25}$$

We need to determine the generating function for the composition $\Phi_{32}\circ\Phi_{21}$,

$$S_{31}(x,r)=f_{31}(x)+\varphi_{31}^\mu(x)r_\mu+\dots. \tag{1.26}$$

(Here the dots stand for the terms of higher order in momenta.) In the lowest order, we have

$$\begin{aligned}S_{31}(x,r)=S_{32}(y,r)+S_{21}(x,q)-y^iq_i&=f_{32}(y)+\varphi_{32}^\mu(y)r_\mu+f_{21}(x)+\varphi_{21}^i(x)q_i-y^iq_i+\dots\\&=f_{32}(y)+\varphi_{32}^\mu(y)r_\mu+f_{21}(x)+\dots=f_{32}(\varphi_{21}(x))+\varphi_{32}^\mu(\varphi_{21}(x))r_\mu+f_{21}(x)+\dots.\end{aligned}$$

Here we are calculating modulo J^2 where the ideal J is generated by the momenta and the zero-order terms such as f_{21}. Note that y^i have to be determined only modulo J, so from the second equation in (1.24) we have $y^i=\varphi_{21}^i(x) \bmod J$, and the terms $\varphi_{21}^i(x)q_i$ and y^iq_i mutually cancel. Therefore, we see that

$$f_{31}=\varphi_{21}^*(f_{32})+f_{21},\qquad \varphi_{31}=\varphi_{32}\circ\varphi_{21}. \tag{1.27}$$

This means that, in the lowest order, we obtain the composition in the semidirect product category $\mathcal{S}\mathcal{M}\mathrm{an}\rtimes\mathbf{C}^\infty$. The objects in this category are supermanifolds and the morphisms are pairs (φ_{21},f_{21}), where $\varphi_{21}\colon M_1\to M_2$ is a supermanifold map and $f_{21}\in\mathbf{C}^\infty(M_1)$ is an even function on the source supermanifold, with the composition of pairs $(\varphi_{32},f_{32})\circ(\varphi_{21},f_{21})=(\varphi_{32}\circ\varphi_{21},\varphi_{21}^*f_{32}+f_{21})$.

Remark 4. The category $\mathcal{S}\mathcal{M}\mathrm{an}\rtimes\mathbf{C}^\infty$ is a closed subspace in the formal category $\mathcal{E}\mathfrak{T}\mathrm{hick}$, and the whole $\mathcal{E}\mathfrak{T}\mathrm{hick}$ is its formal neighborhood. Our calculations show that there are inclusion and retraction functors

$$\mathcal{S}\mathcal{M}\mathrm{an}\rtimes\mathbf{C}^\infty\rightleftarrows\mathcal{E}\mathfrak{T}\mathrm{hick}.$$

Theorem 4. *For pullbacks defined by thick morphisms the identity*

$$(\Phi_{32}\circ\Phi_{21})^*=\Phi_{21}^*\circ\Phi_{32}^* \tag{1.28}$$

holds.

Proof. Consider $f_3\in\mathbf{C}^\infty(M_3)$. Then for $\Phi_{32}^*[f_3]$ we have

$$\Phi_{32}^*[f_3]=f_3+S_{32}-z^\mu r_\mu$$

and for $(\Phi_{21}^*\circ\Phi_{32}^*)[f_3]$ we obtain

$$(\Phi_{21}^*\circ\Phi_{32}^*)[f_3]=\Phi_{21}^*[\Phi_{32}^*[f_3]]=f_3+S_{32}-z^\mu r_\mu+S_{21}-y^iq_i.$$

This coincides with

$$\Phi_{31}^*[f_3] = f_3 + S_{31} - z^\mu r_\mu = f_3 + S_{32} + S_{21} - y^i q_i - z^\mu r_\mu$$

by (1.23), where $\Phi_{31} = \Phi_{32} \circ \Phi_{21}$. $\square$

So far we have dealt with even functions, and what we have defined as $\mathcal{E}\mathcal{T}$hick will be called the *even microformal category*. Parallel constructions are based on the *anticotangent* bundles, i.e., the cotangent bundles with reversed parity in the fibers (see [44]). For local coordinates x^a on a supermanifold M, let x_a^* be the conjugate antimomenta (fiber coordinates on ΠT^*M). The canonical odd symplectic form on ΠT^*M is

$$\omega = d(dx^a x_a^*) = -(-1)^{\widetilde{a}}\, dx^a\, dx_a^* = -(-1)^{\widetilde{a}}\, dx_a^*\, dx^a, \tag{1.29}$$

and let $-\Pi T^*M$ denote ΠT^*M considered with the form $-\omega$.

Definition 4. An *odd thick morphism* (or *odd microformal morphism*) $\Psi\colon M_1 \Rrightarrow M_2$ is specified by a formal odd generating function $S = S(x, y^*)$ (defined locally) and corresponds to a formal canonical relation $\Psi \subset \Pi T^*M_2 \times (-\Pi T^*M_1)$ (denoted by the same letter),

$$\Psi = \left\{ (y^i, y_i^*; x^a, x_a^*) \;\Big|\; y^i = \frac{\partial S}{\partial y_i^*}(x, y^*),\ x_a^* = \frac{\partial S}{\partial x^a}(x, y^*) \right\}. \tag{1.30}$$

On the submanifold Ψ we have

$$dy^i y_i^* - dx^a x_a^* = d(y^i y_i^* - S). \tag{1.31}$$

Under changes of coordinates, the odd generating function S of an odd thick morphism has the transformation law

$$S'(x', y'^*) = S(x, y^*) - y^i y_i^* + y^{i'} y_{i'}^* \tag{1.32}$$

similar to (1.8), where variables on the right-hand side are determined from the equations similar to those that arise in the even case.

The following Theorems 5–7 are completely analogous to the “even” versions above, and we omit their proofs.

Theorem 5. *There is a well-defined composition $\Psi_{32} \circ \Psi_{21}$ of odd thick morphisms, which is an odd thick morphism*

$$\Psi_{31}\colon\ M_1 \Rrightarrow M_3$$

with the generating function $S_{31} = S_{31}(x, z^*)$, where

$$S_{31}(x, z^*) = S_{32}(y, z^*) + S_{21}(x, y^*) - y^i y_i^* \tag{1.33}$$

and y^i and y_i^* are expressed uniquely via (x^a, z_μ^*) from the system

$$y_i^* = \frac{\partial S_{32}}{\partial y^i}(y, z^*) \qquad \text{and} \qquad y^i = \frac{\partial S_{21}}{\partial y_i^*}(x, y^*) \tag{1.34}$$

as a power series in z_μ^* and a functional power series in S_{32}. $\square$

Theorem 6. *The composition of odd thick morphisms is associative.* $\square$

Odd thick morphisms form a formal category $\mathcal{O}\mathcal{T}$hick, which we call the *odd microformal category*. It is the formal neighborhood of the subcategory $\mathcal{S}\mathcal{M}$an $\rtimes$ $\mathbf{\Pi C}^\infty$ contained as a closed subspace (and there are inclusion and retraction functors). The affine action of the category $\mathcal{S}\mathcal{M}$an $\rtimes$ $\mathbf{\Pi C}^\infty$ on supermanifolds of odd functions extends to a nonlinear action of the formal category $\mathcal{O}\mathcal{T}\mathrm{hick}$ as follows.

Definition 5. The *pullback* Ψ^* with respect to an odd thick morphism $\Psi\colon M_1 \Rrightarrow M_2$ is a formal mapping of functional supermanifolds

$$\Psi^*\colon\ \mathbf{\Pi C}^{\infty}(M_2) \to \mathbf{\Pi C}^{\infty}(M_1) \tag{1.35}$$

defined for $\gamma \in \mathbf{\Pi C}^{\infty}(M_2)$ by

$$\Psi^*[\gamma](x) = \gamma(y) + S(x, y^*) - y^i y_i^*, \tag{1.36}$$

where y_i^* and y^i are determined from the equations

$$y_i^* = \frac{\partial \gamma}{\partial y^i}(y) \qquad \text{and} \qquad y^i = \frac{\partial S}{\partial y_i^*}(x, y^*). \tag{1.37}$$

Theorem 7. *For odd thick morphisms, the identity*

$$(\Psi_{32} \circ \Psi_{21})^* = \Psi_{21}^* \circ \Psi_{32}^* \tag{1.38}$$

holds. $\square$

As in the even case, the pullback Ψ^* is a formal nonlinear differential operator for which the kth term in the power expansion contains derivatives of orders $\leq k-1$. An analog of Theorem 1 holds [44]. One can formulate "odd" versions of Definition 3 and Conjecture 1.

Remark 5. Pullback of functions with respect to a thick morphism is a particular case of the composition of thick morphisms (both in the bosonic and fermionic cases)—the same as for usual pullbacks. One may wish to consider "thick functions" on supermanifolds as thick morphisms to $\mathbb{R}$ or $\mathbb{C}$. One may also wish to consider gluing "thick supermanifolds" from ordinary ones with the help of thick diffeomorphisms or, for example, to introduce "thick analogs" of Lie groups. Constructions in this section suggest many attractive paths, which we hope to explore in the future.

2. APPLICATION TO VECTOR BUNDLES: THE NOTION OF THE ADJOINT FOR A NONLINEAR MAP

In this section, we generalize the notion of the adjoint of a linear operator. We show that using thick morphisms one can speak of the *adjoint* for a nonlinear map of vector spaces or vector bundles. Such generalized adjoints are thick morphisms rather than ordinary maps. There are two versions of this construction, "even" and "odd."

Our construction is based on the canonical diffeomorphism between the cotangents of dual vector bundles discovered by Kirill Mackenzie and Ping Xu [28, Theorem 5.5][6] (see also [26; 27, Ch. 9] and [36] for the super case):

$$T^*E \cong T^*E^*, \tag{2.1}$$

which will be referred to as the Mackenzie–Xu transformation. (Some authors use the name "Legendre transformation," but this is really confusing since the Legendre transformation or transform in the standard sense acts on functions, not points.) There is a parallel canonical diffeomorphism for the fermionic case [36]

$$\Pi T^*E \cong \Pi T^*(\Pi E^*). \tag{2.2}$$

[6]The special case of $E = TM$, i.e., the diffeomorphism $T^*TM \cong T^*T^*M$, is due to Tulczyjew [33]; the case of general E was considered independently by J.-P. Dufour in an unpublished work.

Recall these natural diffeomorphisms in the form suitable for our purposes. For a vector bundle $E \to M$, denote local coordinates on the base by x^a and linear coordinates in the fibers by u^i. The transformation law for u^i has the form $u^i = u^{i'} T_{i'}{}^i$. Denote the fiber coordinates for the dual bundle $E^* \to M$ and the antidual bundle $\Pi E^* \to M$ by u_i and η_i, respectively. We assume that the invariant bilinear forms are $u^i u_i$ and $u^i \eta_i$. (This means that u_i and η_i are the right coordinates with respect to the basis which is "right dual" to a basis in E.) Consider the cotangent and the anticotangent bundles for E. Denote the canonically conjugate momenta for x^a and u^i by p_a and p_i, and the conjugate antimomenta, by x^*_a and u^*_i. A similar notation will be used for E^* and ΠE^*.

The *Mackenzie–Xu transformation*

$$\boldsymbol{\kappa}\colon\, T^*E \to T^*E^* \tag{2.3}$$

is defined by the formulas

$$\boldsymbol{\kappa}^*(x^a) = x^a, \qquad \boldsymbol{\kappa}^*(u_i) = p_i, \qquad \boldsymbol{\kappa}^*(p_a) = -p_a, \qquad \boldsymbol{\kappa}^*(p^i) = (-1)^{\tilde{i}} u^i. \tag{2.4}$$

It is well defined and is an antisymplectomorphism. (The choice of signs in (2.4) agrees with that in the book [27] and differs from that of [36]. The choice used in [36] gives a symplectomorphism.)

An odd version of this transformation [36] (which we denote by the same letter)

$$\boldsymbol{\kappa}\colon\, \Pi T^*E \to \Pi T^*(\Pi E^*) \tag{2.5}$$

is defined by

$$\boldsymbol{\kappa}^*(x^a) = x^a, \qquad \boldsymbol{\kappa}^*(\eta_i) = u^*_i, \qquad \boldsymbol{\kappa}^*(x^*_a) = -x^*_a, \qquad \boldsymbol{\kappa}^*(\eta^{*i}) = u^i \tag{2.6}$$

(note the absence of signs depending on parities). It is also an antisymplectomorphism with respect to the canonical odd symplectic structures.

Remark 6. The invariance of formulas (2.4), (2.6) is nontrivial and follows from the analysis of T^*E and ΠT^*E as *double vector bundles* over M. On the other hand, from the coordinate formulas (2.4) and (2.6), it is obvious that $\boldsymbol{\kappa}^*\omega = -\omega$ for the canonical symplectic structures. Moreover, one can immediately see that for the canonical Liouville 1-forms

$$\boldsymbol{\kappa}^*(dx^a p_a + du_i p^i) = -(dx^a p_a + du^i p_i) + d(u^i p_i) \tag{2.7}$$

on the cotangent bundle and

$$\boldsymbol{\kappa}^*(dx^a x^*_a + d\eta_i \eta^{*i}) = -(dx^a x^*_a + du^i u^*_i) + d(u^i u^*_i) \tag{2.8}$$

on the anticotangent bundle. Note that $u^i p_i$ and $u^i u^*_i$ are invariant functions.

Now we proceed to construct generalized adjoints. Let E_1 and E_2 be vector bundles over a fixed base M. Consider a fiber map over M,

$$\Phi\colon\, E_1 \to E_2,$$

which is not necessarily fiberwise linear. (Here Φ is an ordinary map, not a thick morphism.) In local coordinates, it is given by

$$\Phi^*(y^a) = x^a, \qquad \Phi^*(w^\alpha) = \Phi^\alpha(x, u)$$

for some functions $\Phi^\alpha(x, u)$, where u^i and w^α are linear coordinates on the fibers of E_1 and E_2. For the fiber coordinates on the dual bundles we use the same letters with the lower indices so that the forms $u^i u_i$ and $w^\alpha w_\alpha$ give invariant pairings.

Note that it makes sense to speak about fiberwise thick morphisms.

Theorem 8. 1. *For an arbitrary fiberwise map of vector bundles $\Phi\colon E_1 \to E_2$ over a base M, there are a fiberwise even thick morphism* ("adjoint")

$$\Phi^*\colon\ E_2^* \Rrightarrow E_1^* \tag{2.9}$$

and a fiberwise odd thick morphism ("antiadjoint")

$$\Phi^{*\Pi}\colon\ \Pi E_2^* \Rightarrow \Pi E_1^* \tag{2.10}$$

such that if the map $\Phi\colon E_1 \to E_2$ is fiberwise linear, i.e., is a vector bundle homomorphism, then the thick morphisms Φ^* and $\Phi^{*\Pi}$ are ordinary maps which are the usual adjoint homomorphism and the adjoint homomorphism combined with the parity reversion, respectively.

2. *For the composition of fiberwise maps of vector bundles over M,*

$$E_1 \xrightarrow{\Phi_{21}} E_2 \xrightarrow{\Phi_{32}} E_3, \tag{2.11}$$

we have the equality

$$(\Phi_{32}\circ\Phi_{21})^* = \Phi_{21}^*\circ\Phi_{32}^* \tag{2.12}$$

of even thick morphisms $E_3^* \Rrightarrow E_1^*$ and the equality

$$(\Phi_{32}\circ\Phi_{21})^{*\Pi} = \Phi_{21}^{*\Pi}\circ\Phi_{32}^{*\Pi} \tag{2.13}$$

of odd thick morphisms $\Pi E_3^* \Rightarrow \Pi E_1^*$.

Proof. Consider a fiberwise map[7] $\Phi\colon E_1 \to E_2$,

$$(x^a, u^i) \mapsto \bigl(y^a = x^a,\ w^\alpha = \Phi(x^a, u^i)\bigr).$$

To the map Φ corresponds the canonical relation $R_\Phi \subset T^*E_2 \times (-T^*E_1)$,

$$R_\Phi = \bigl\{(y^a, w^\alpha, q_i, q_\alpha; x^a, u^i, p_a, p_i) \bigm| (-1)^{\widetilde{a}}\, dq_a\, y^a + (-1)^{\widetilde{\alpha}}\, dq_\alpha\, w^\alpha + dx^a\, p_a + du^i\, p_i = dS\bigr\},$$

with the generating function $S = S(x^a, u^i, q_i, q_\alpha)$, where

$$S = x^a q_a + \Phi^\alpha(x^a, u^i) q_\alpha. \tag{2.14}$$

We define the thick morphism $\Phi^*\colon E_2^* \Rrightarrow E_1^*$ by a generating function $S^* = S^*(y^a, w_\alpha, p_a, p^i)$, where

$$S^* := y^a p_a + \Phi^\alpha\bigl(y^a, (-1)^{\widetilde{i}} p^i\bigr) w_\alpha. \tag{2.15}$$

The corresponding canonical relation $\Phi^* \subset T^*E_1^* \times (-T^*E_2^*)$ is given by the equation

$$(-1)^{\widetilde{a}}\, dp_a\, x^a + (-1)^{\widetilde{i}}\, dp^i\, u_i + dy^a\, q_a + dw_\alpha\, q^\alpha = dS^*,$$

or, more explicitly,

$$x^a = y^a, \qquad u_i = \frac{\partial\Phi^\alpha}{\partial u^i}\bigl(y, (-1)^{\widetilde{i}} p^i\bigr) w_\alpha,$$

$$q_a = p_a + \frac{\partial\Phi^\alpha}{\partial x^a}\bigl(y, (-1)^{\widetilde{i}} p^i\bigr) w_\alpha, \qquad q^\alpha = (-1)^{\widetilde{\alpha}}\Phi^\alpha\bigl(y, (-1)^{\widetilde{i}} p^i\bigr).$$

The construction of the thick morphism Φ^* can be stated geometrically as follows. We first apply the transformation $\boldsymbol{\kappa}$ to the canonical relation $R_\Phi \subset T^*E_2 \times (-T^*E_1)$. Since $\boldsymbol{\kappa}$ is an antisymplectomorphism, we obtain a Lagrangian submanifold $(\boldsymbol{\kappa}\times\boldsymbol{\kappa})(R_\varphi) \subset -T^*E_2^* \times T^*E_1^*$. The thick

[7] For clarity, although against our own taste, we use the physicists' notation with arguments of functions equipped with indices.

morphism Φ^* is then defined by the opposite relation:

$$\Phi^* := \big((\boldsymbol{\kappa}\times\boldsymbol{\kappa})(R_\varphi)\big)^{\mathrm{op}} \subset T^*E_1^* \times (-T^*E_2^*).$$

Expressing this by generating functions, we arrive at the formulas above. One can see that the thick morphism $\Phi^*\colon E_2^* \to E_1^*$ is also fiberwise. Let us check that Φ^* is the ordinary adjoint when $\Phi\colon E_1\to E_2$ is linear on fibers. Indeed, in such a case we have

$$\Phi(x^a,u^i) = u^i {\Phi_i}^\alpha(x);$$

hence the above formulas give

$$x^a = y^a, \qquad u_i = {\Phi_i}^\alpha(y) w_\alpha,$$

as expected. The odd thick morphism $\Phi^{*\Pi}\colon \Pi E_2^* \Rightarrow \Pi E_1^*$ is built in a similar way: we take the canonical relation $R_\Phi\subset \Pi T^*E_2\times(-\Pi T^*E_1)$ corresponding to a map $\Phi\colon E_1\to E_2$, apply the odd version of the Mackenzie–Xu transformation and then take the opposite relation.

To obtain equations (2.12) and (2.13), notice that the composition of maps (2.11) induces the composition of the corresponding canonical relations between the cotangent bundles in the same order. This is preserved by the Mackenzie–Xu transformation. Taking the opposite relations reverses the order. $\square$

Corollary 1. *On functions on the dual bundles, the pullback with respect to the adjoint $\Phi^*\colon E_2^* \to E_1^*$ induces a* "nonlinear pushforward map"

$$\Phi_* := (\Phi^*)^*\colon\ \mathbf{C}^\infty(E_1^*)\to \mathbf{C}^\infty(E_2^*). \tag{2.16}$$

The restriction of Φ_* to the space of even sections $\mathbf{C}^\infty(M,E_1)$ regarded as the subspace in $\mathbf{C}^\infty(E_1^*)$ consisting of the fiberwise linear functions takes it to the subspace $\mathbf{C}^\infty(M,E_2)$ in $\mathbf{C}^\infty(E_2^*)$ and coincides with the usual pushforward of sections $\Phi_*(\boldsymbol{v}) = \Phi\circ\boldsymbol{v}$.

Proof. The nonlinear pushforward $\Phi_*\colon \mathbf{C}^\infty(E_1^*)\to\mathbf{C}^\infty(E_2^*)$ is defined as the pullback with respect to the thick morphism $\Phi^*\colon E_2^*\to E_1^*$. To an even function $f=f(x^a,u_i)$ the map Φ_* assigns the even function $g=\Phi_*[f]$, where $g(x^a,w_\alpha)$ is given by

$$g(x,w_\alpha) = f(x,u_i) + \Phi^\alpha\big(x,(-1)^{\tilde{i}}p^i\big)w_\alpha - u_ip^i, \tag{2.17}$$

and u_i and p^i are found from the equations

$$p^i = \frac{\partial f}{\partial u_i}(x,u_i) \qquad\text{and}\qquad u_i = \frac{\partial\Phi^\alpha}{\partial u^i}\left(x,(-1)^{\tilde{i}}\,\frac{\partial f}{\partial u_i}(x,u_i)\right)w_\alpha. \tag{2.18}$$

The latter equation is solvable by iterations. Now let the function f on E_1^* have the form $f(x,u_i)=v^i(x)u_i$, which corresponds to an even section $\boldsymbol{v}=v^i(x)\boldsymbol{e}_i$ of the bundle E_1. Then

$$p^i = (-1)^{\tilde{i}}v^i(x); \tag{2.19}$$

hence

$$\Phi_*[f] = v^i(x)u_i + \Phi^\alpha(x,v^i(x))w_\alpha - u_i(-1)^{\tilde{i}}v^i(x) = \Phi^\alpha(x,v^i(x))w_\alpha, \tag{2.20}$$

which is the fiberwise linear function on E_2^* corresponding to the section $\Phi\circ\boldsymbol{v}$. $\square$

A similar statement holds for the odd case: there is an *odd nonlinear pushforward* map $\Phi_*^\Pi := (\Phi^{*\Pi})^*$,

$$\Phi_*^\Pi\colon\ \boldsymbol{\Pi}\mathbf{C}^\infty(\Pi E_1^*)\to\boldsymbol{\Pi}\mathbf{C}^\infty(\Pi E_2^*). \tag{2.21}$$

On the space of even sections $\boldsymbol{v} \in \mathbf{C}^{\infty}(M, E_1)$ regarded as a subspace of fiberwise linear functions $\mathbf{C}^{\infty}(M, E_1) \subset \mathbf{\Pi C}^{\infty}(\Pi E_1^*)$ in the space of all odd functions on ΠE_1^*, the map Φ_*^{Π} again coincides with the obvious pushforward $\boldsymbol{v} \mapsto \Phi \circ \boldsymbol{v}$.

The algebra of fiberwise polynomial functions on the dual bundle E^* is freely generated by the sections of E over the algebra of functions on the base M. For the vector bundle homomorphisms $E_1 \to E_2$, the pushforward of functions $C^{\infty}(E_1^*) \to C^{\infty}(E_2^*)$ is the algebra homomorphism extending a linear map from free generators. As seen from Corollary 1, the nonlinear pushforward map $\Phi_*\colon \mathbf{C}^{\infty}(E_1^*) \to \mathbf{C}^{\infty}(E_2^*)$ can be similarly regarded as the extension of a "nonlinear homomorphism" from generators.

Remark 7. If the base M is a point, we have a nonlinear map of vector spaces $\Phi\colon V \to W$. Replacing it by the Taylor expansion gives a sequence of linear maps $\Phi_k\colon S^kV \to W$. The functions on the dual spaces can themselves be seen as elements of the symmetric powers. By expanding the pushforward Φ_* in a Taylor series, we arrive at linear maps of the form $S^n(\bigoplus S^pV) \to \bigoplus S^qW$. It would be interesting to obtain for them a purely algebraic description.

Remark 8. From the proof of Theorem 8 it is clear that instead of an ordinary map one can start from a fiberwise even thick morphism $E_1 \Rrightarrow E_2$ and construct its adjoint $E_2^* \Rrightarrow E_1^*$ by the same method; or one can start from a fiberwise odd thick morphism $E_1 \Rightarrow E_2$ and construct the antiadjoint $\Pi E_2^* \Rightarrow \Pi E_1^*$.

Remark 9. "Nonlinear adjoints" can be generalized to vector bundles over different bases by using the concept of comorphisms of Higgins and Mackenzie [16].[8] Suppose $E_1 \to M_1$ and $E_2 \to M_2$ are fiber bundles over bases M_1 and M_2. Then a *bundle morphism* $\Phi\colon E_1 \to E_2$ can be defined as a fiberwise map over the fixed base $E_1 \to \varphi^*E_2$ and a *bundle comorphism* $\Psi\colon E_1 \to E_2$ can be defined as a fiberwise map over the fixed base $\psi^*E_1 \to E_2$. (It would be better to use arrows of different shape for morphisms and comorphisms.) In both cases, there is a map of the bases φ or ψ pointing in the same direction for a morphism and in the opposite direction for a comorphism.

For bundles over the same base, morphisms and comorphisms over the identity map coincide, and for manifolds regarded as "zero vector bundles," morphisms are ordinary maps while comorphisms are morphisms in the opposite category. As was shown in [16], for vector bundles (assuming the fiberwise linearity for maps over a fixed base), the adjoint of a morphism $E_1 \to E_2$ is a comorphism $E_2^* \to E_1^*$ and vice versa; so this gives an anti-isomorphism of the two categories of vector bundles. To generalize this to our setup, one may wish to keep a map between the bases as an ordinary map while using fiberwise thick morphisms over a fixed base. This incorporates the possible nonlinearity of morphisms. In this way, one obtains base-changing "thick morphisms" and "thick comorphisms" of vector bundles to which the duality theory extends.

3. APPLICATION TO LIE ALGEBROIDS AND HOMOTOPY POISSON BRACKETS

It is well known that, for Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a linear map of the underlying vector spaces $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism if the adjoint map of the dual spaces $\varphi^*\colon \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ is Poisson with respect to the induced Lie–Poisson brackets (also known as the Berezin–Kirillov brackets). The same holds true for Lie algebroids [27, Ch. 10] (see [16] for base-changing morphisms). In this section we use the construction of the adjoint for a nonlinear map of vector bundles and the results from [44] to establish the homotopy analogs of these statements for the case of L_∞-morphisms of L_∞-algebroids. It is convenient to work in the super setting, though we generally suppress the prefix "super-."

[8] This notion has a rich prehistory and numerous connections. Besides citations in [16], see Guillemin and Sternberg [15], who suggested redefining morphisms of vector bundles as, basically, comorphisms. A close notion was introduced in [35] in connection with integral transforms. In [7] it is argued that comorphisms are the "correct" notion in the context of Poisson geometry.

For simplicity consider the case of fixed base. We do not consider the “if and only if” form of the statement. Our main theorem here is as follows.

Theorem 9. *An L_∞-morphism of L_∞-algebroids over a base M induces L_∞-morphisms of the homotopy Poisson and homotopy Schouten algebras of functions on the dual and antidual bundles, respectively.*

Before proving the theorem, we recall some definitions and statements.

Recall that an L_∞-*algebroid* (see, e.g., [21]) is a (super) vector bundle $E \to M$ endowed with a sequence of n-ary *brackets* that defines an L_∞-algebra structure on sections and a sequence of n-ary *anchors* $a\colon E \times_M \ldots \times_M E \to TM$ (multilinear bundle maps) so that the brackets satisfy the Leibniz identities with respect to the multiplication of sections by functions on the base,

$$[u_1, \ldots, u_{n-1}, f u_n] = a(u_1, \ldots, u_{n-1})(f)\, u_n + (-1)^{(\widetilde{u}_1+\ldots+\widetilde{u}_{n-1}+n)\widetilde{f}} f\, [u_1, \ldots, u_n]. \tag{3.1}$$

Here we follow the convention of Lada and Stasheff for L_∞-algebras [25] that the brackets are antisymmetric and of alternating parities. So the unary bracket is odd, the binary bracket is even, etc. (Under the alternative convention, all brackets are symmetric and odd. Its equivalence with the antisymmetric convention is by the parity reversion; see the discussion in [37]. In the sequel, we shall need to use both versions.) With this convention, ordinary Lie algebroids are a particular case of L_∞-algebroids. An L_∞-algebroid structure on $E \to M$ is equivalent to a formal homological vector field on the supermanifold ΠE. An L_∞-*morphism* of L_∞-algebroids $\Phi\colon E_1 \rightsquigarrow E_2$ is specified by a fiberwise map (in general, nonlinear) $\Phi\colon \Pi E_1 \to \Pi E_2$ such that the corresponding homological vector fields are Φ-related.[9] With some abuse of language, it is convenient to call the map $\Pi E_1 \to \Pi E_2$ itself an L_∞-morphism. This definition includes as particular cases L_∞-morphisms of L_∞-algebras and morphisms of Lie algebroids. Note that what we call L_∞-algebras are often called “curved” L_∞-algebras. By default we include a 0-ary operation.

An L_∞-algebroid structure on $E \to M$ induces a homotopy Poisson structure on the supermanifold E^* and a homotopy Schouten structure on the supermanifold ΠE^*. This means that there are given sequences of brackets turning the space $C^\infty(E^*)$ into an L_∞-algebra in the Lada–Stasheff sense (“antisymmetric convention”) and $C^\infty(\Pi E^*)$ into an L_∞-algebra in the sense of the alternative (“symmetric”) convention. Each bracket must also be a derivation in each argument. We shall refer to these brackets as the *homotopy Lie–Poisson* and *homotopy Lie–Schouten* brackets. These structures on E^* and ΠE^*, as well as the homological vector field on ΠE, are all equivalent to each other and should be seen as different manifestations of one structure of an L_∞-algebroid, as in the familiar cases of Lie algebras and Lie algebroids [36, 40].

Proof of Theorem 9. Consider an L_∞-algebroid $E \to M$. We shall give the proof for the homotopy Lie–Schouten brackets on ΠE^*. (The case of the homotopy Lie–Poisson brackets on E^* is similar.) Let Q_E be the homological vector field on ΠE specifying the algebroid structure in E. The homotopy Lie–Schouten brackets of functions on ΠE^* are the higher derived brackets generated by the odd *master Hamiltonian* $H^* = H^*_E$ (i.e., an odd function on the cotangent bundle $T^*(\Pi E^*)$ satisfying $(H^*, H^*) = 0$ for the canonical Poisson bracket), which is obtained from the fiberwise linear Hamiltonian $H_E = Q_E \cdot p$ on $T^*(\Pi E)$ by the Mackenzie–Xu diffeomorphism $T^*(\Pi E) \cong T^*(\Pi E^*)$. Suppose there is an L_∞-morphism of L_∞-algebroids $E_1 \rightsquigarrow E_2$, i.e., a map $\Phi\colon \Pi E_1 \to \Pi E_2$ over M such that the vector fields Q_1 and Q_2 are Φ-related. This is equivalent to the Hamiltonians $H_1 = H_{E_1}$ and $H_2 = H_{E_2}$ being R_Φ-related [44, Sect. 2, Example 6]. By applying the Mackenzie–Xu transformations and flipping the factors, we conclude that the Hamiltonians $H^*_2 = H^*_{E_2}$ and $H^*_1 = H^*_{E_1}$ are Φ^*-related, where $\Phi^*\colon \Pi E^*_2 \Rrightarrow \Pi E^*_1$ is the adjoint thick morphism. By a key statement from [44]

[9] Note that here there is *no* single map of manifolds from E_1 to E_2; hence the nonstandard arrow $E_1 \rightsquigarrow E_2$ denoting a morphism.

(corollary to Theorems 6 and 7), if the master Hamiltonians are related by a thick morphism, then the pullback is an L_∞-morphism of the homotopy Schouten algebras of functions. Hence the pushforward map $\Phi_* = (\Phi^*)^*\colon \mathbf{C}^\infty(\Pi E_1^*) \to \mathbf{C}^\infty(\Pi E_2^*)$ is an L_∞-morphism, as claimed. $\square$

With suitable modifications, the statement should hold for base-changing morphisms.

The following lemma should be known. It extends the corresponding property of ordinary Lie algebroids [27]. We give a proof for completeness (compare with the statement for higher Lie algebroids [38, 39]).

Lemma 1. *For an L_∞-algebroid $E \to M$, the higher anchors assemble into an L_∞-morphism*

$$a\colon\ E \rightsquigarrow TM,$$

to which we also refer as an anchor (*and for which we use the same notation*), *where TM has the standard Lie algebroid structure.*

Proof. The sequence of n-ary anchors assembles into a single map $a\colon \Pi E \to \Pi TM$, which is given by $a = \Pi Tp \circ Q$, where $Q = Q_E$ and $\Pi Tp\colon \Pi T(\Pi E) \to \Pi TM$ is the differential of the bundle projection $p\colon \Pi E \to M$. For an arbitrary Q-manifold N, the map $Q\colon N \to \Pi TN$ is tautologically a Q-morphism; i.e., the vector fields Q on N and d on ΠTN are Q-related. Also, for any map, its differential is a Q-morphism of the antitangent bundles. Hence the map $a\colon \Pi E \to \Pi TM$ is a Q-morphism as the composition of Q-morphisms. Therefore, it gives an L_∞-morphism $E \rightsquigarrow TM$ (which we denote by the same letter). $\square$

Corollary 2. *The anchor for every L_∞-algebroid $E \to M$ induces an L_∞-morphism*

$$a_*\colon\ \mathbf{C}^\infty(\Pi E^*) \to \mathbf{C}^\infty(\Pi T^*M) \tag{3.2}$$

for the homotopy Lie–Schouten brackets and an L_∞-morphism

$$a_*\colon\ \mathbf{\Pi C}^\infty(E^*) \to \mathbf{\Pi C}^\infty(T^*M) \tag{3.3}$$

for the homotopy Lie–Poisson brackets. (*The functions on the bundles ΠT^*M and T^*M are considered with the canonical Schouten and Poisson brackets, respectively.*)

Note that on the right-hand sides of (3.2) and (3.3) there is only a binary bracket, while on the left-hand sides there are in general infinitely many brackets with all numbers of arguments. Therefore, for a general L_∞-algebroid $E \to M$, these L_∞-morphisms must be nontrivial, i.e., expressed by supermanifold maps that are substantially nonlinear.

Corollary 3. *On a homotopy Poisson manifold M, there is an L_∞-morphism*

$$\mathbf{C}^\infty(\Pi TM) \to \mathbf{C}^\infty(\Pi T^*M), \tag{3.4}$$

where functions on ΠTM (i.e., pseudodifferential forms) *are considered with the higher Koszul brackets introduced in* [21].

To appreciate the meaning of Corollary 3, recall that for an ordinary Poisson structure on a (super)manifold M, there is a linear transformation from forms to multivector fields, $\Omega^k(M) \to \mathfrak{A}^k(M)$, preserving degrees and parities, basically "raising indices" with the help of the Poisson tensor, which intertwines the de Rham differential on forms and the Poisson–Lichnerowicz differential on multivector fields, as well as the Koszul bracket on forms and the Schouten bracket on multivector fields. Recall that the Poisson–Lichnerowicz differential d_P can be defined by $d_P = [\![P, -]\!]$, the Schouten bracket with the Poisson tensor. The Koszul bracket induced by a Poisson structure can be defined on 1-forms by formulas such as $[df, dg]_P = d\{f, g\}_P$ and $[df, g]_P = \{f, g\}_P$, where $\{f, g\}_P$ is a given Poisson bracket, and then extended to all forms as a biderivation. It is best to see this as a Lie algebroid structure induced on T^*M (see [27]). For the homotopy case, the picture will be as follows [21]. A single binary Koszul bracket is replaced by an infinite sequence of "higher Koszul brackets" on $\Omega(M)$ making T^*M an L_∞-algebroid. It is still possible to define a linear transformation from forms to multivectors (no longer preserving degrees), such that the diagram

$$\begin{array}{ccc} \mathfrak{A}(M) & \xrightarrow{d_P} & \mathfrak{A}(M) \\ \uparrow & & \uparrow \\ \Omega(M) & \xrightarrow{d} & \Omega(M) \end{array}$$

is commutative, where the analog of the Poisson–Lichnerowicz differential $d_P = [\![P, -]\!]$ is an odd operator (but not of a particular degree). However, there is a problem with the brackets. Unlike the classical case, this linear map $\Omega(M) \to \mathfrak{A}(M)$ (and any linear map) clearly cannot transform a sequence of many higher Koszul brackets into one Schouten bracket. We conjectured in [21] that an L_∞-morphism from $\Omega(M)$ to $\mathfrak{A}(M)$ must exist instead. Corollary 3 gives the desired solution. The linear map from forms to multivectors constructed in [21] is induced by a fiberwise (nonlinear) map $\Pi T^*M \to \Pi TM$, which represents the anchor $T^*M \rightsquigarrow TM$. The dual to it is a thick morphism $\Pi T^*M \Rrightarrow \Pi TM$, the nonlinear pullback by which is exactly the sought L_∞-morphism. See [22] for details.

Corollary 4 (generalization of Corollary 3)**.** *There is an L_∞-morphism of homotopy Schouten algebras*

$$\mathbf{C}^\infty(\Pi E) \to \mathbf{C}^\infty(\Pi E^*) \tag{3.5}$$

for "triangular L_∞-bialgebroids."

Recall that Mackenzie and Xu [28] introduced the concept of a *triangular Lie bialgebroid* as a generalization of Drinfeld's triangular Lie bialgebras. It is a pair of vector bundles in duality (E, E^*), where a Lie algebroid structure on E is initially given and the bundle E^* is made a Lie algebroid with the help of an element $r \in \Gamma(M, \Lambda^2 E)$ playing the role of the classical r-matrix. In our language, r is a fiberwise quadratic function on ΠE^*. The Lie algebroid structure on E^* is defined by the Hamiltonian $H_{E^*} := (H_E^*, r) \in C^\infty(T^*(\Pi E^*))$, where H_E^* is obtained by the Mackenzie–Xu transformation from the Hamiltonian $H_E \in C^\infty(T^*(\Pi E))$ corresponding to the Lie algebroid structure on E. (Counting weights shows that the Hamiltonian H_{E^*} is linear in momenta on ΠE^*, as required.) The pair (TM, T^*M) for a Poisson manifold is a model example of a triangular Lie bialgebroid. The role of an r-matrix is played by the Poisson bivector. Transporting this analogy to the homotopy case, we can define an L_∞-analog of the Mackenzie–Xu triangular Lie bialgebroids. For a pair (E, E^*), one starts from an L_∞-algebroid structure on E and an even function r on ΠE^* (no constraints on degrees), and then introduces a compatible "triangular" structure, which will make the pair (E, E^*) a *triangular L_∞-bialgebroid.*[10] The key observation here is that the homotopy analog of a triangular structure is the shift in the argument of the master Hamiltonian, $H(x,p) \mapsto H'(x,p) = H\big(x, p + \frac{\partial r}{\partial x}\big)$. Corollary 4 in this setting arises as an abstract version of Corollary 3. We elaborate these questions elsewhere (see [22]; another paper is in preparation[11]).

[10] There is some freedom as to what should be called an L_∞-*bialgebroid* structure on (E, E^*) in general. The options range from L_∞-algebroid structures on E and E^* with a compatibility condition expressible as $(H_1, H_2) = 0$ for the corresponding odd Hamiltonians which live on $T^*(\Pi E) \cong T^*(\Pi E^*)$, which in particular gives a self-commuting Hamiltonian $H := H_1 + H_2$, to the apparently more general structure described by a single self-commuting odd Hamiltonian H of an arbitrary form. In the latter option the distinction between the bialgebroid and its "Drinfeld double" looks blurred. One should certainly wish to have a pair of structures that can be combined into a family.

[11] Th. Th. Voronov, "L_∞-bialgebroids and homotopy Poisson structures" (in preparation).

4. QUANTUM THICK MORPHISMS: GENERAL PROPERTIES

We shall show now that the construction of thick morphisms in the bosonic case has a "quantum" counterpart. Namely, we shall define "quantum pullbacks" depending on Planck's constant $\hbar$ as certain oscillatory integral operators that transform functions on one (super)manifold to functions on another (super)manifold. We then define the "quantum microformal category" as the dual to the category of such integral operators. We shall show that in the limit $\hbar \to 0$ this picture gives rise to thick morphisms and the corresponding nonlinear pullbacks. This in hindsight may be seen as clarifying the origin of the above "classical" constructions. For quantum pullbacks it is possible to give a closed formula as opposed to the pullbacks by "classical" thick morphisms, defined only by an iterative procedure. Quantum thick morphisms were first introduced in the short note [42]. Some further results were obtained in the preprint [41].

We first need to introduce a class of functions on which quantum pullbacks will be acting.

Definition 6. An *oscillatory wave function* on a (super)manifold M is a linear combination of formal expressions of the form

$$w_\hbar(x) = a(\hbar, x)e^{\frac{i}{\hbar}b(x,\hbar)} \tag{4.1}$$

where $a(x,\hbar) = \sum_{n\geq 0} \hbar^n a_n(x)$ and $b(x,\hbar) = \sum_{n\geq 0} \hbar^n b_n(x)$ are formal expansions in nonnegative powers of $\hbar$ whose coefficients are smooth functions on M. (Here $b(x,\hbar)$ is even.)

We customarily drop explicit indication of dependence on $\hbar$ for oscillatory wave functions and write $w(x)$ for $w_\hbar(x)$. We assume natural rules of manipulations with the expression like (4.1). Note that we can always rearrange the exponential in (4.1) so as to make $w(x) = A(\hbar, x)e^{\frac{i}{\hbar}b_0(x)}$, with no dependence on $\hbar$ in the phase $b_0(x)$. Conversely, an invertible factor in front of the exponential can be made into a term in the phase if we forsake the reality restriction. Oscillatory wave functions on M form an algebra, which we denote by $OC^\infty_\hbar(M)$ and which extends the algebra $C^\infty_\hbar(M) := C^\infty(M)[[\hbar]]$ of formal power series in $\hbar$ with smooth coefficients. Symbolically,

$$OC^\infty_\hbar(M) = C^\infty_\hbar(M)\exp\frac{i}{\hbar}C^\infty(M).$$

Consider supermanifolds M_1 and M_2. In the same way as thick morphisms $M_1 \Rrightarrow M_2$ are specified by their generating functions, quantum thick morphisms will be specified by certain "quantum" generating functions. As in Section 1, denote by x^a local coordinates on M_1, by y^i local coordinates on M_2, and by p_a and q_i the corresponding conjugate momenta. In given coordinate systems on M_1 and M_2, a *quantum generating function* $S_\hbar(x,q)$ is a formal power series in q_i,

$$S_\hbar(x,q) = S^0_\hbar(x) + \varphi^i_\hbar(x)q_i + \frac{1}{2}S^{ij}_\hbar(x)q_jq_i + \frac{1}{3!}S^{ijk}_\hbar(x)q_kq_jq_i + \dots, \tag{4.2}$$

where the coefficients are formal power series in $\hbar$. Note that, the same as for the "classical" case considered before, $S_\hbar(x,q)$ is a coordinate representation of a geometric object and not a scalar function. Its transformation law will be clarified shortly.

Definition 7. A *quantum thick morphism*, or *quantum microformal morphism*,

$$\widehat{\Phi}\colon\ M_1 \Rrightarrow_q M_2$$

with a (quantum) generating function $S_\hbar(x,q)$ is identified with its action on functions

$$\widehat{\Phi}^*\colon\ OC^\infty_\hbar(M_2) \to OC^\infty_\hbar(M_1)$$

in the opposite direction, called *quantum pullback* and defined by the formula

$$(\widehat{\Phi}^*w)(x) = \int\limits_{T^*M_2} Dy\, \mathit{Đ}q\, e^{\frac{i}{\hbar}(S_\hbar(x,q) - y^iq_i)}\, w(y). \tag{4.3}$$

Integration in (4.3) is with respect to the normalized Liouville measure $Dy\,\bar{D}q$ on T^*M_2. Here and in the future, we use the notation $\bar{D}q := (2\pi\hbar)^{-n}(i\hbar)^m Dq$ if Dq is a coordinate volume element in (super)dimension $n|m$.

The source of the normalization factor above is in the formulas for the direct and inverse $\hbar$-Fourier transform. Recall that on $\mathbb{R}^{n|m}$ they read

$$\widetilde{f}(p) = \int Dx\; e^{-\frac{i}{\hbar}x^a p_a} f(x)$$

and

$$f(x) = (2\pi\hbar)^{-n}(i\hbar)^m \int Dp\; e^{\frac{i}{\hbar}x^a p_a}\widetilde{f}(p) = \int \bar{D}p\; e^{\frac{i}{\hbar}x^a p_a}\widetilde{f}(p),$$

where the integration is over $\mathbb{R}^{n|m}$ and over its dual. (There may be an extra common sign factor depending on choices of signs in the Berezin integral, which we take to be 1.) In particular,

$$\delta(x) = \int \bar{D}p\; e^{\frac{i}{\hbar}x^a p_a}.$$

Remark 10. The form of integral operators (4.3) is familiar in the theory of partial differential equations (not the super case of course). Operators of a slightly more general form

$$Au(x) = \int e^{i(S(x,p)-x'p)} a(x,p)\, u(x')\, dx'\,đp, \tag{4.4}$$

but with both x and x' in the same domain $\Omega \subset \mathbb{R}^n$, were studied by M. I. Vishik and G. I. Eskin [34] and especially by Yu. V. Egorov [8, 9] and M. V. Fedoryuk [11]. Together with Maslov's canonical operator [29], they were precursors of the Fourier integral operators introduced by L. Hörmander in [17, 18]. Hörmander [18] stressed as a crucial observation of Egorov a connection between operators (4.4) and canonical transformations in $T^*\Omega$. (As noticed in [11], such a connection was indicated earlier by V. A. Fock [12], who was making a precise statement out of Dirac's analogy between unitary transformations in quantum mechanics and canonical transformations in classical mechanics. See also [13, Pt. I, Ch. III, § 16].) In Hörmander's construction of Fourier integral operators, canonical transformations gave way to canonical relations, specified by equivalence classes of phase functions depending on auxiliary variables. In standard theory, these canonical relations are conical, so the phase functions are positively homogeneous of degree +1 (see [32, 10, 31]). Operators (4.3) can therefore be seen as a special case of Fourier integral operators, but not exactly fitting in the standard definitions because of the different type of their phase functions.

Example 8. Let $S_\hbar(x,q) = S^0_\hbar(x) + \varphi^i_\hbar(x)q_i$. Then

$$(\widehat{\Phi}^*w)(x) = e^{\frac{i}{\hbar}S^0_\hbar(x)} \int\limits_{T^*M_2} \bar{D}(y,q)\; e^{\frac{i}{\hbar}(\varphi^i_\hbar(x)-y^i)q_i}\, w(y) = e^{\frac{i}{\hbar}S^0_\hbar(x)} w(\varphi_\hbar(x)). \tag{4.5}$$

We arrive at a "quantum analog" of the category $\mathcal{SM}\mathrm{an} \rtimes \mathbf{C}^\infty$ and its action on smooth functions. Morphisms here are pairs $(\varphi, e^{\frac{i}{\hbar}f})$, and the composition of pairs is given by

$$\bigl(\varphi_{32}, e^{\frac{i}{\hbar}f_{32}}\bigr) \circ \bigl(\varphi_{21}, e^{\frac{i}{\hbar}f_{21}}\bigr) = \bigl(\varphi_{32}\circ\varphi_{21}, e^{\frac{i}{\hbar}(\varphi^*_{21}f_{32}+f_{21})}\bigr).$$

The phase functions f and maps φ are expansions in nonnegative powers of $\hbar$, $f = f_\hbar$ and $\varphi = \varphi_\hbar$ (so the "maps" are formal perturbations of ordinary maps). The action (4.5) is clearly well defined for oscillatory wave functions w.

Example 9. Let $w(y) \equiv 1$. Then, for arbitrary $S_\hbar(x, q)$, we have

$$\widehat{\Phi}^*(1) = e^{\frac{i}{\hbar} S^0_\hbar(x)}. \tag{4.6}$$

Example 10. Let $w(y) = e^{\frac{i}{\hbar} yc}$, where $yc \equiv y^i c_i$ and c_i are parameters. Then

$$(\widehat{\Phi}^* w)(x) = e^{\frac{i}{\hbar} S_\hbar(x,c)} \tag{4.7}$$

(cf. Example 6). We can restate this as a formula for reconstructing the quantum generating function:

$$e^{\frac{i}{\hbar} S_\hbar(x,q)} = \widehat{\Phi}^*\big[e^{\frac{i}{\hbar} yq}\big](x) \tag{4.8}$$

(where we restored q in the argument).

Let $S_0(x, q)$ be obtained by substituting $\hbar = 0$ in a quantum generating function $S_\hbar(x, q)$. We regard $S_0(x, q)$ as the generating function of a classical thick morphism $\Phi\colon M_1 \Rrightarrow M_2$. Before we have clarified the transformation law for quantum generating functions, this would make sense at least in a fixed coordinate system. We shall write $\Phi = \lim_{\hbar\to 0} \widehat{\Phi}$.

Theorem 10. *In the limit $\hbar \to 0$, the quantum pullback $\widehat{\Phi}^*$ transforms the phase of an oscillatory wave function as the pullback Φ^* by the classical thick morphism $\Phi = \lim_{\hbar\to 0} \widehat{\Phi}$, so that if $w(y) = e^{\frac{i}{\hbar} g(y)}$ on M_2, then on M_1*

$$(\widehat{\Phi}^* w)(x) = e^{\frac{i}{\hbar} f_\hbar(x)}, \qquad \textit{where} \quad f_\hbar = \Phi^*[g] + O(\hbar).$$

Proof. For a wave function $w(y) = e^{\frac{i}{\hbar} g(y)}$, we have

$$(\widehat{\Phi}^* w)(x) = \int\limits_{T^*M_2} \mathit{Đ}(y, q)\; e^{\frac{i}{\hbar}(S_\hbar(x,q) - y^i q_i + g(y))}.$$

By the stationary phase method (see Appendix A), the value of the integral, in the main order in $\hbar$, is the exponential evaluated at the critical points of the phase when $\hbar \to 0$. By differentiating with respect to y^i and q_i and setting the result to zero, we arrive at the system of equations

$$q_i = \frac{\partial g}{\partial y^i}(y), \qquad y^i = (-1)^{\tilde{\imath}}\, \frac{\partial S_0}{\partial q_i}(x, q)$$

for determining y^i and q_i, whose unique solution should be substituted into $S_0(x, q) + g(y) - y^i q_i$ to obtain a function $f(x)$ as the leading term of the phase. These are exactly equations (1.15) in the definition of pullback, and $f = \Phi^*[g]$ as claimed. $\square$

Remark 11. The stationary phase method [11] can be applied to $\widehat{\Phi}^* w$ for $w = a(x, \hbar)e^{\frac{i}{\hbar} g(x)}$, and it also allows to find all terms in the expansion in $\hbar$ (at least, their general form), not only the main term. The fact that quantum pullback preserves the class of oscillatory wave functions follows from here. Note that the square root of the Hessian arising as a factor in the stationary phase method can be formally subsumed into the phase as a correction of the first order in $\hbar$. Note also that since in the main order the quantum pullback reduces to the classical pullback, which is a formal map, so is the quantum pullback (formal on the phases). For convenience, we included the precise statements concerning the stationary phase method in the form suitable for our needs in the Appendix (see Theorems 17 and 18 there).

The integral (4.3) can actually be solved in a closed form, giving an expression for a quantum pullback $\widehat{\Phi}^*\colon OC^\infty_\hbar(M_2) \to OC^\infty_\hbar(M_1)$ as a "formal differential operator." (This is an advantage over pullbacks by classical thick morphisms, given in general only by an iterative procedure.) Let us write a quantum generating function $S_\hbar(x,q)$ defining a quantum microformal morphism $\widehat{\Phi}\colon M_1 \to_q M_2$ in the form similar to (1.12),

$$S_\hbar(x,q) = S^0_\hbar(x) + \varphi^i_\hbar(x)q_i + S^+_\hbar(x,q), \tag{4.9}$$

where $S^+_\hbar(x,q)$ is the sum of all terms of order ≥ 2 in q_i.

Theorem 11. *The action of $\widehat{\Phi}^*\colon OC^\infty_\hbar(M_2) \to OC^\infty_\hbar(M_1)$ can be expressed as follows:*

$$(\widehat{\Phi}^* w)(x) = e^{\frac{i}{\hbar}S^0_\hbar(x)} \Big(e^{\frac{i}{\hbar}S^+_\hbar\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)} w(y)\Big)\Big|_{y^i=\varphi^i_\hbar(x)}. \tag{4.10}$$

It is a formal differential operator over a map $\varphi_\hbar\colon M_1 \to M_2$ given by $y^i = \varphi^i_\hbar(x)$.

Proof. Substituting (4.9) into (4.3) gives

$$\begin{aligned}(\widehat{\Phi}^* w)(x) &= \int \text{Đ}(y,q)\; e^{\frac{i}{\hbar}(S^0_\hbar(x)+\varphi^i_\hbar(x)q_i+S^+_\hbar(x,q)-y^iq_i)}\, w(y)\\ &= e^{\frac{i}{\hbar}S^0_\hbar(x)} \int \text{Đ}q\; e^{\frac{i}{\hbar}\varphi^i_\hbar(x)q_i} e^{\frac{i}{\hbar}S^+_\hbar(x,q)} \int Dy\; e^{-\frac{i}{\hbar}y^iq_i}\, w(y).\end{aligned}$$

The integral is the composition of the ($\hbar$-)Fourier transform of a function $w(y)$ from the variables y^i to the variables q_i, the multiplication by $e^{\frac{i}{\hbar}S^+_\hbar(x,q)}$, treated as a function of q_i with x^a seen as parameters, and the inverse Fourier transform from q_i to y^i, where $\varphi^i_\hbar(x)$ is substituted for y^i, followed finally by the multiplication by the phase factor $e^{\frac{i}{\hbar}S^0_\hbar(x)}$. Recalling the standard relation between multiplication and differentiation under Fourier transform, we arrive at the claimed result. ☐

Remark 12. The notion of a differential operator over a smooth map as opposed to operators on a single manifold is not very standard, but should be self-explanatory. Separating the "differentiation part" such as $S^+_\hbar\big(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\big)$ from the purely "substitution part" $y^i = \varphi^i_\hbar(x)$ in (4.10) is of course coordinate-dependent. Naively, there are three ingredients in $\widehat{\Phi}^*$: a differential operator of infinite order in y^i and of the form $1 + O(\hbar)$ in $\hbar$ (starting with the second derivatives and where each term with the derivatives of order k is of order $k-1$ in $\hbar$), the substitution as such, and the multiplication by the phase factor. Thus, a general quantum thick morphism $\widehat{\Phi}$ can be seen as a perturbation, due to the term $S^+_\hbar(x,q)$ in the expansion (4.9) of the generating function, of a morphism of the form $(\varphi_\hbar, e^{\frac{i}{\hbar}f_\hbar})$ as in Example 8.

We can push this a bit further by noticing that the quantum pullback $\widehat{\Phi}^*$ can be written as an integral operator

$$(\widehat{\Phi}^* w)(x) = \int Dy\, K(x,y)\, w(y) \tag{4.11}$$

with the Schwarz kernel

$$K(x,y) = \int \text{Đ}q\; e^{\frac{i}{\hbar}(S_\hbar(x,q)-y^iq_i)}, \tag{4.12}$$

i.e., the $\hbar$-Fourier transform (up to a factor) of the function $e^{\frac{i}{\hbar}S_\hbar(x,q)}$ from q to y. By expanding $S_\hbar(x,q)$ as in (4.9) and using manipulations similar to those in the proof of Theorem 11, we can express the integral kernel of the operator $\widehat{\Phi}^*$ as

$$K(x,y) = e^{\frac{i}{\hbar}S^0_\hbar(x)} e^{\frac{i}{\hbar}S^+_\hbar\left(x,-\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}\, \delta(y-\varphi_\hbar(x)) \tag{4.13}$$

(note the minus sign in the argument of $S^{+}_{\hbar}$). This is basically a restatement of Theorem 11. In this form it is clear that the integral kernel of $\widehat{\Phi}^*$ is supported on a formal neighborhood of the graph of the "$\hbar$-perturbed" map $\varphi_\hbar\colon M_1 \to M_2$.

Theorem 12. *The composition of quantum thick morphisms $M_1 \xrightarrow{\widehat{\Phi}_{21}}_q M_2 \xrightarrow{\widehat{\Phi}_{32}}_q M_3$ with generating functions $S_{21}(x_1,p_2)$ and $S_{32}(x_2,p_3)$ is a quantum thick morphism $M_1 \xrightarrow{\widehat{\Phi}_{31}}_q M_3$ with the generating function $S_{31}(x_1,p_3)$ given by*

$$e^{\frac{i}{\hbar}S_{31}(x_1,p_3)} = \int\limits_{T^*M_2} \mathit{Đ}(x_2,p_2)\, e^{\frac{i}{\hbar}(S_{32}(x_2,p_3)+S_{21}(x_1,p_2)-x_2p_2)}. \tag{4.14}$$

(Here $S_{21} := S_{21,\hbar}$, etc.; we suppress $\hbar$ to simplify the notation.) *In the limit $\hbar\to 0$, this composition law becomes the composition law for classical thick morphisms given by Theorem 2.*

Proof. Apply the composition $\widehat{\Phi}^*_{21}\circ\widehat{\Phi}^*_{32}$ to a "test function" $w(x_3)=e^{\frac{i}{\hbar}x_3p_3}$ (see Example 10). The claim is that the result is an oscillatory exponential of the desired form. We work in the abbreviated notation and denote coordinates on the manifolds M_i by x_i and the conjugate momenta by p_i, where $i=1,2,3$. We have

$$\begin{aligned}
\widehat{\Phi}^*_{21}\big(\widehat{\Phi}^*_{32}\big[e^{\frac{i}{\hbar}x_3p_3}\big]\big)(x_1) &= \widehat{\Phi}^*_{21}\big[\widehat{\Phi}^*_{32}\big[e^{\frac{i}{\hbar}x_3p_3}\big](x_2)\big](x_1)\\
&= \int Dx_2\, \mathit{Đ}p_2\, e^{\frac{i}{\hbar}(S_{21}(x_1,p_2)-x_2p_2)} \int Dx_3\, \mathit{Đ}p'_3\, e^{\frac{i}{\hbar}(S_{32}(x_2,p'_3)-x_3p'_3)} e^{\frac{i}{\hbar}x_3p_3}\\
&= \int Dx_2\, \mathit{Đ}p_2\, Dx_3\, \mathit{Đ}p'_3\, e^{\frac{i}{\hbar}(S_{21}(x_1,p_2)+S_{32}(x_2,p'_3)-x_2p_2+x_3(p_3-p'_3))}\\
&= \int Dx_2\, \mathit{Đ}p_2\, e^{\frac{i}{\hbar}(S_{21}(x_1,p_2)+S_{32}(x_2,p_3)-x_2p_2)}.
\end{aligned}$$

From the stationary phase method (see Theorem 18 in the Appendix) we observe, first, that the latter integral can be written as an exponential $e^{\frac{i}{\hbar}S_{31}(x_1,p_3)}$ for some function S_{31} depending on $\hbar$ and, second, that in the limit $\hbar\to 0$, which is indicated by 0 in the subscripts, we should have

$$S_{31,0}(x_1,p_3) = S_{21,0}(x_1,p_2) + S_{32,0}(x_2,p_3) - x_2p_2,$$

where the variables x_2 and p_2 are found from the equations

$$x_2^i = (-1)^{\tilde{\imath}}\,\frac{\partial S_{21,0}}{\partial p_{2\,i}}(x_1,p_2), \qquad p_{2\,i} = \frac{\partial S_{32,0}}{\partial x_2^i}(x_2,p_3).$$

This is exactly the composition law for classical generating functions as given by Theorem 2. $\square$

Theorem 13 (transformation law for quantum generating functions)**.** *Let $x^a = x^a(x')$, $y^i = y^i(y')$ and $x^{a'} = x^{a'}(x)$, $y^{i'} = y^{i'}(y)$ be mutually inverse changes of local coordinates on $M_1\times M_2$. Then quantum generating functions $S_\hbar(x,q)$ and $S'_\hbar(x',q')$ specifying the same quantum thick morphism $\widehat{\Phi}\colon M_1 \to_q M_2$ in the coordinate systems x,y and x',y' are related by the transformation law*

$$e^{\frac{i}{\hbar}S'_\hbar(x',q')} = \int Dy\, \mathit{Đ}q\, e^{\frac{i}{\hbar}(S_\hbar(x(x'),q)-yq+y'(y)q')}, \tag{4.15}$$

where we use abbreviated notation such as $yq\equiv y^iq_i$.

Proof. Similarly to the proof of Theorem 12, apply $\widehat{\Phi}^*$, for a quantum thick morphism $\widehat{\Phi}$ specified by $S_\hbar(y,q)$ in the "old" coordinates x^a, y^i, to a test function $w = e^{\frac{i}{\hbar}y^{i'}q_{i'}}$, where $y^{i'}$ are the "new" coordinates on M_2 and $q_{i'}$ are the conjugate momenta, expressing the result also via the "new" coordinates $x^{a'}$ on M_1. We obtain

$$\widehat{\Phi}^*\bigl[e^{\frac{i}{\hbar}y'q'}\bigr] = \int Dy\, \text{Đ}q\; e^{\frac{i}{\hbar}(S(x,q)-yq)}\, e^{\frac{i}{\hbar}y'(y)q'} = \int Dy\, \text{Đ}q\; e^{\frac{i}{\hbar}(S(x,q)-yq+y'(y)q')},$$

where it remains to substitute $x = x(x')$. The integral is of the type covered by Theorem 18 in the Appendix, and we may conclude that it equals to an oscillating exponential of the form $e^{\frac{i}{\hbar}S'_\hbar(x',q')}$, which therefore gives the quantum generating function of the morphism $\widehat{\Phi}$ in the "new" coordinates on M_1 and M_2 expressed by (4.15), as claimed. $\square$

(This included the independence of the notion of a quantum thick morphism of a choice of coordinates.)

If we apply the stationary phase method to the integral on the right-hand side of (4.15), we shall arrive at the equations

$$y^i = (-1)^{\tilde{\imath}}\,\frac{\partial S_\hbar}{\partial q_i}(x(x'),q), \qquad q_i = \frac{\partial y^{i'}}{\partial y^i}(y)q_{i'}$$

for determining y^i and q_i (as functions of x' and q') at the stationary point. Then

$$S'_\hbar(x',q') = S_\hbar(x(x'),q) - yq + y'(y)q' + O(\hbar).$$

Hence, in the limit $\hbar \to 0$, the transformation law for quantum generating functions $S_\hbar$ becomes, as anticipated, the transformation law (1.8) for classical generating functions S, $S = S_0$, considered before.

Remark 13. For quantum thick morphisms, there are two different kinds of power expansions: the expansion in Planck's constant $\hbar$ and the expansions already present for classical thick morphisms (formal power expansions for the pullbacks and compositions), which can be compared with expansions "in the coupling constant." The source of latter is the higher order terms in momenta in generating functions, which in particular result in coupled equations for determining the stationary phase points. See also Appendix A.

5. QUANTUM THICK MORPHISMS: APPLICATION TO HOMOTOPY ALGEBRAS

Now we turn to application of quantum microformal morphisms to homotopy bracket structures. Since the initial motivation for introducing "classical" microformal morphisms was the search for a construction of L_∞-morphisms for homotopy Poisson or Schouten brackets, it is natural to ask about the respective position of the quantum version.

For the "quantum" context we need to recall how a bracket structure is generated by a differential operator. Let A be a commutative associative superalgebra with unit. Suppose Δ is a linear operator acting on A. One can say when Δ is a differential operator (d.o.) of order (less than or equal to) n. This is defined by induction: Δ is of order 0 if it commutes with multiplication by elements of A, and of order n if for all $a \in A$ the commutator $[\Delta, a]$ is of order $n-1$. (By using Hadamard's lemma, one can see that for a smooth manifold this leads to the usual definition with partial derivatives.) Such an understanding can be traced back to A. Grothendieck [14, Ch. IV, § 16.8]. J.-L. Koszul [24] extracted from it a construction of a sequence of multilinear operations (later generalized by F. Akman from commutative to other algebras; see, e.g., [1]), which we shall call "brackets,"[12] and which are defined as follows: for an arbitrary linear operator Δ on an algebra A and for elements $a_1, \ldots, a_k \in A$, where $k \geq 0$, set

$$\{a_1, \ldots, a_k\}_\Delta := [\ldots[\Delta, a_1], \ldots, a_k](1). \tag{5.1}$$

[12] Hopefully, no confusion will arise with the Koszul brackets on differential forms considered in Section 3.

For $k = 0, 1, 2, 3$ one can find

$$
\begin{aligned}
\{\varnothing\}_\Delta &= \Delta(1),\\
\{a\}_\Delta &= \Delta(a) - \Delta(1)a,\\
\{a,b\}_\Delta &= \Delta(ab) - \Delta(a)b - (-1)^{\widetilde{a}\widetilde{b}}\Delta(b)a + \Delta(1)ab,\\
\{a,b,c\}_\Delta &= \Delta(abc) - \Delta(ab)c - (-1)^{\widetilde{b}\widetilde{c}}\Delta(ac)b - (-1)^{\widetilde{a}(\widetilde{b}+\widetilde{c})}\Delta(bc)a\\
&\quad + \Delta(a)bc + (-1)^{\widetilde{a}\widetilde{b}}\Delta(b)ac + (-1)^{(\widetilde{a}+\widetilde{b})\widetilde{c}}\Delta(c)ab - \Delta(1)abc,
\end{aligned}
$$

and an expression of this form can be written for arbitrary k (see below). Koszul's construction is an example of "higher derived brackets" [37]. The brackets are symmetric in the super sense and have parity equal to the parity of Δ. For any k, they satisfy the identity

$$
\begin{aligned}
\{a_1,\dots,a_{k-1},a_ka_{k+1}\}_\Delta &= \{a_1,\dots,a_{k-1},a_k\}_\Delta a_{k+1} + (-1)^{\alpha_k}a_k\{a_1,\dots,a_{k-1},a_{k+1}\}_\Delta\\
&\quad + \{a_1,\dots,a_{k-1},a_k,a_{k+1}\}_\Delta, \qquad (5.2)
\end{aligned}
$$

where $\alpha_k = \widetilde{a}_k(\widetilde{\Delta} + \widetilde{a}_1 + \ldots + \widetilde{a}_{k-1})$, which means that the $(k+1)$th bracket measures the failure of the kth bracket to be a derivation in its arguments. If Δ is a differential operator of order n, then all brackets with more than n arguments vanish, the top bracket is a multiderivation, and in the formula for it there is no need to evaluate at 1,

$$\{a_1,\dots,a_n\}_\Delta = [\dots[\Delta,a_1],\dots,a_n].$$

The top bracket can be identified with the *principal symbol* of a differential operator. We refer to the operator Δ as the *generating operator* of the sequence of brackets $\{-,\dots,-\}_\Delta$.

Remark 14. For arbitrary k, the expression for the kth bracket generated by Δ is

$$\{a_1,\dots,a_k\}_\Delta = \sum_{s=0}^{k}(-1)^s \sum_{(k-s,\,s)\text{-shuffles}} (-1)^\alpha\,\Delta\big(a_{\tau(1)}\dots a_{\tau(k-s)}\big)\,a_{\tau(k-s+1)}\dots a_{\tau(k)}, \qquad (5.3)$$

where $(-1)^\alpha = (-1)^{\alpha(\tau;\widetilde{a}_1,\dots,\widetilde{a}_k)}$ is the standard "Koszul sign" for permutation of commuting factors of given parities. (If all elements $a_1,\dots,a_k$ are even, then $(-1)^{\alpha(\tau;\widetilde{a}_1,\dots,\widetilde{a}_k)} = 1$.)

If Δ is odd, the brackets are also odd and one may ask about their Jacobiators. As shown in [37], the sequence of the Jacobiators is generated by the operator $\Delta^2 = \frac{1}{2}[\Delta,\Delta]$. In particular, if $\Delta^2 = 0$, all the Jacobiators vanish and the brackets generated by Δ make A an L_∞-algebra (in the symmetric version).[13]

Note however that we do not obtain an S_∞-algebra (or "homotopy Schouten" algebra) in this way because the Leibniz identity is not satisfied. Following [37], we can modify Koszul's construction to resolve this problem. Consider $A_\hbar := A[[\hbar]]$. Define *$\hbar$-differential operators* ($\hbar$-d.o.'s) as follows. Let Δ be a linear operator on $A_\hbar$. Then Δ is an $\hbar$-d.o. of order 0 if it commutes with the multiplication by all $a \in A_\hbar$, and Δ is an $\hbar$-d.o. of order n if for all $a \in A_\hbar$ the operator $[\Delta,a]$ has the form $(-i\hbar)\Delta'_a$, where Δ'_a is an $\hbar$-d.o. of order $n-1$. For example, if Δ' is a d.o. of order n in the usual sense, then the operator $\Delta = (-i\hbar)^n\Delta'$ is an $\hbar$-d.o. of order n.

Example 11. On a (super)manifold M, an arbitrary $\hbar$-d.o. of order n has the form

$$\Delta = (-i\hbar)^n A_\hbar^{a_1\dots a_n}(x)\,\partial_{a_1}\dots\partial_{a_n} + (-i\hbar)^{n-1}A_\hbar^{a_1\dots a_{n-1}}(x)\,\partial_{a_1}\dots\partial_{a_{n-1}} + \ldots + A_\hbar^0(x). \qquad (5.4)$$

[13] This was first found in physics literature related to the Batalin–Vilkovisky formalism (see, e.g., [3]).

We note that the algebra of oscillatory wave functions introduced above is stable under $\hbar$-d.o.'s. (It is not stable under arbitrary differential operators because they may create the factors of $\hbar^{-1}$.)

For an $\hbar$-d.o. Δ of arbitrary order, all k-fold commutators $[\ldots[\Delta, a_1], \ldots, a_k]$ are divisible by $(-i\hbar)^k$, and we can (re)define the brackets generated by Δ by setting

$$\{a_1, \ldots, a_k\}_{\Delta,\hbar} := (-i\hbar)^{-k}[\ldots[\Delta, a_1], \ldots, a_k](1). \tag{5.5}$$

We can also introduce the corresponding "classical" brackets by

$$\{a_1, \ldots, a_k\}_{\Delta,0} := \lim_{\hbar\to 0}(-i\hbar)^{-k}[\ldots[\Delta, a_1], \ldots, a_k](1) \tag{5.6}$$

(the limit has the meaning of substituting 0 in the nonnegative power expansion in $\hbar$). We refer to (5.5) as the quantum brackets as opposed to the classical brackets (5.6). The quantum brackets satisfy the identity

$$\begin{aligned}\{a_1, \ldots, a_{k-1}, a_k a_{k+1}\}_{\Delta,\hbar} = \{a_1, \ldots, a_{k-1}, a_k\}_{\Delta,\hbar}\, a_{k+1} + (-1)^{\widetilde{\alpha}} a_k \{a_1, \ldots, a_{k-1}, a_{k+1}\}_{\Delta,\hbar} \\ + (-i\hbar)\{a_1, \ldots, a_{k-1}, a_k, a_{k+1}\}_{\Delta,\hbar},\end{aligned} \tag{5.7}$$

which for $\hbar \to 0$ becomes the derivation property

$$\{a_1, \ldots, a_{k-1}, a_k a_{k+1}\}_{\Delta,0} = \{a_1, \ldots, a_{k-1}, a_k\}_{\Delta,0}\, a_{k+1} + (-1)^{\widetilde{\alpha}} a_k \{a_1, \ldots, a_{k-1}, a_{k+1}\}_{\Delta,0}. \tag{5.8}$$

Here $\widetilde{\alpha} = \widetilde{a}_k(\widetilde{\Delta} + \widetilde{a}_1 + \ldots + \widetilde{a}_{k-1})$. We call the sequence of all classical brackets generated by Δ the *principal symbol* of an $\hbar$-d.o. Δ. On a (super)manifold, since the classical brackets are symmetric multiderivations of the algebra of functions, the principal symbol can be identified with an inhomogeneous polynomial in momentum variables. (In the language of Section 3, it is the master Hamiltonian of the brackets.)

Example 12. For an $\hbar$-d.o. of Example 11, the principal symbol is

$$H(x,p) = A_0^{a_1\ldots a_n}(x)\, p_{a_1}\ldots p_{a_n} + A_0^{a_1\ldots a_{n-1}}(x)\, p_{a_1}\ldots p_{a_{n-1}} + \ldots + A_0^0(x), \tag{5.9}$$

which is an inhomogeneous fiberwise polynomial function on T^*M, well-defined independently of a choice of coordinates! (The subscript 0 means substituting 0 for $\hbar$ in the coefficients.)

Remark 15. If we only keep the condition that all k-fold commutators $[\ldots[\Delta, a_1], \ldots, a_k]$ be divisible by $(-i\hbar)^k$, formula (5.5) still makes sense and we obtain, generally, an infinite sequence of brackets. We shall refer to such operators as *formal $\hbar$-differential operators.* On manifolds, this gives operators whose principal symbols are formal power series in momenta. Algebraic constructions here agree with the known notion of $\hbar$-pseudodifferential operators defined in local coordinates by integrals

$$(\Delta u)(x) = \iint Đp\, Dx'\, e^{\frac{i}{\hbar}(x^a - x'^a)p_a} H_\hbar(x,p)\, u(x'),$$

with a function $H_\hbar(x,p)$ from a suitable symbol class (see, e.g., [31]). Here the "full symbol" $H_\hbar(x,p)$ is coordinate-dependent, but the principal symbol $H(x,p) = H_0(x,p)$ is well defined as a function on T^*M.

Suppose an odd operator Δ squares to 0. Consider the quantum brackets (5.5). They define an L_∞-algebra (in the odd symmetric version) and additionally satisfy the modified Leibniz identity (5.7). We shall call such an algebraic structure an *$S_{\infty,\hbar}$-algebra* (so that for $\hbar = 0$ we come back to an S_∞-algebra, $S_{\infty,0} = S_\infty$). We shall give a formula for the corresponding homological vector field, as well as a formula for the master Hamiltonian for the classical S_∞-algebra (i.e., the principal symbol of Δ).

Lemma 2. *The quantum brackets* (5.5) *correspond to a formal vector field Q on an algebra A* (*more accurately, on the corresponding supermanifold* $\mathbf{A}$), *where*

$$Q = e^{-\frac{i}{\hbar}a}\Delta\bigl(e^{\frac{i}{\hbar}a}\bigr)\frac{\delta}{\delta a}. \tag{5.10}$$

Here $a\in A$ and $\Delta(e^{\frac{i}{\hbar}a})$ denotes the application of the operator to the function.

Proof. The formal vector field corresponding to a sequence of symmetric multilinear functions of fixed parity on a superspace A is the formal sum

$$Q(a) = \sum_{k=0}^{+\infty}\frac{1}{k!}\{\underbrace{a,\dots,a}_{k}\}$$

(see, e.g., [37]). Here $a\in A$ is a "running" even element (or a point of the corresponding supermanifold). A vector field here is identified with a vector function. It can also be expressed as

$$Q = \sum_{k=0}^{+\infty}\frac{1}{k!}\{\underbrace{a,\dots,a}_{k}\}\frac{\delta}{\delta a},$$

meaning an infinitesimal shift $a\mapsto a+\varepsilon Q(a)$. The relation of the vector field Q to the given multilinear functions is by the higher derived bracket formula [37]

$$\{a_1,\dots,a_k\} = [\dots[Q,a_1],\dots,a_k](0)$$

(the value of a vector field at the origin). Here the vectors a_i are regarded as constant vector fields. Now, to obtain (5.10), we take an even element a in the algebra A and consider

$$\{\underbrace{a,\dots,a}_{k}\}_{\Delta,\hbar} = \left[\dots\left[\Delta,\frac{i}{\hbar}a\right],\dots,\frac{i}{\hbar}a\right](1) = \left(\left(\operatorname{ad}\left(-\frac{i}{\hbar}a\right)\right)^k\Delta\right)(1);$$

hence

$$\begin{aligned}Q(a) &= \sum_{k=0}^{+\infty}\frac{1}{k!}\left(\left(\operatorname{ad}\left(-\frac{i}{\hbar}a\right)\right)^k\Delta\right)(1) = \left(e^{\operatorname{ad}(-\frac{i}{\hbar}a)}\Delta\right)(1) = \bigl(\operatorname{Ad}\bigl(e^{-\frac{i}{\hbar}a}\bigr)\Delta\bigr)(1)\\ &= \bigl(e^{-\frac{i}{\hbar}a}\Delta e^{\frac{i}{\hbar}a}\bigr)(1) = e^{-\frac{i}{\hbar}a}\Delta\bigl(e^{\frac{i}{\hbar}a}(1)\bigr) = e^{-\frac{i}{\hbar}a}\Delta\bigl(e^{\frac{i}{\hbar}a}\bigr). \qquad\square\end{aligned}$$

Lemma 3. *In the differential-geometric setting, the principal symbol of Δ, or the master Hamiltonian of the classical brackets* (5.6), *is given by*

$$H(x,p) = \lim_{\hbar\to 0} e^{-\frac{i}{\hbar}x^a p_a}\Delta\bigl(e^{\frac{i}{\hbar}x^a p_a}\bigr). \tag{5.11}$$

Proof. Recall that the master Hamiltonian H of symmetric brackets is defined by the relation [37]

$$\{f_1,\dots,f_k\} = (\dots(H,f_1),\dots,f_k)\big|_M$$

for functions $f_i\in C^\infty(M)$. Hence, in local coordinates,

$$H(x,p) = \sum_{k=0}^{+\infty}\frac{1}{k!}\{x^{a_1}p_{a_1},\dots,x^{a_k}p_{a_k}\},$$

where the momentum variables p_a are treated as parameters when the brackets are taken; therefore, for the classical brackets generated by an operator Δ, we obtain

$$H(x,p)=\lim_{\hbar\to 0}\sum_{k=0}^{+\infty}\frac{1}{k!}\left[\dots\left[\Delta,\frac{i}{\hbar}x^{a_1}p_{a_1}\right],\dots,\frac{i}{\hbar}x^{a_k}p_{a_k}\right](1)=\lim_{\hbar\to 0}e^{-\frac{i}{\hbar}x^a p_a}\Delta\big(e^{\frac{i}{\hbar}x^a p_a}\big)$$

(where we have effectively repeated the argument used in the proof of Lemma 2). □

Remark 16. For neither formula (5.10) nor (5.11) it is important that the operator Δ generating the brackets is odd or satisfies $\Delta^2=0$. In particular, it makes sense to consider "L_∞-morphisms" of the infinite sequences of brackets generated by arbitrary operators Δ without such assumptions. It is interesting what such morphisms would mean in the classical context of partial differential equations.

Now we shall give a "quantum analog" of Theorem 6 of [44], which says that Poisson thick morphisms induce L_∞-morphisms of homotopy Poisson brackets.

Definition 8. We say that M is a *Batalin–Vilkovisky manifold*, or shortly a *BV-manifold*, if M is a supermanifold equipped with an odd formal $\hbar$-differential operator Δ of square zero. The operator Δ is referred to as the *BV-operator*. For BV-manifolds M_1 and M_2 with BV-operators Δ_1 and Δ_2, we say that a quantum thick morphism $\widehat{\Phi}\colon M_1 \to_q M_2$ is a *quantum BV-morphism* if

$$\Delta_1\circ\widehat{\Phi}^*=\widehat{\Phi}^*\circ\Delta_2. \tag{5.12}$$

The BV-operator on a BV-manifold M specifies an $S_{\infty,\hbar}$-structure on the algebra $C^\infty_\hbar(M_1)$. We shall show that a quantum BV-morphism $\widehat{\Phi}\colon M_1 \to_q M_2$ induces an L_∞-morphism of the corresponding $S_{\infty,\hbar}$-algebras. Note that it cannot be the pullback $\widehat{\Phi}^*$ itself, since $\widehat{\Phi}^*$ is linear and we are looking for a nonlinear map of the function supermanifolds

$$\mathbf{C}^\infty_\hbar(M_2)\to\mathbf{C}^\infty_\hbar(M_1).$$

For a quantum thick morphism $\widehat{\Phi}$ (not necessarily a BV-morphism), define $\widehat{\Phi}^!$ by

$$\widehat{\Phi}^!(g):=\frac{\hbar}{i}\ln\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big) \tag{5.13}$$

for a $g\in\mathbf{C}^\infty_\hbar(M_2)$. If we introduce the notation $\exp_\hbar g:=\exp\big(\frac{i}{\hbar}g\big)$ and $\ln_\hbar f:=\frac{\hbar}{i}\ln f$, then

$$\widehat{\Phi}^!=\ln_\hbar\circ\widehat{\Phi}^*\circ\exp_\hbar. \tag{5.14}$$

For the composition of quantum thick morphisms

$$M_1\xrightarrow{\widehat{\Phi}_{21}}{}_q\, M_2\xrightarrow{\widehat{\Phi}_{32}}{}_q\, M_3,$$

we have

$$(\widehat{\Phi}_{32}\circ\widehat{\Phi}_{21})^!=\widehat{\Phi}_{21}^!\circ\widehat{\Phi}_{32}^!.$$

Theorem 14. *If $\widehat{\Phi}\colon M_1\to_q M_2$ is a quantum BV-morphism, then $\widehat{\Phi}^!$ is an L_∞-morphism of the $S_{\infty,\hbar}$-algebras of functions. In greater detail, the map*

$$\widehat{\Phi}^!\colon\ \mathbf{C}^\infty_\hbar(M_2)\to\mathbf{C}^\infty_\hbar(M_1)$$

is a morphism of Q-manifolds, where the homological vector fields $Q_i\in\operatorname{Vect}(\mathbf{C}^\infty_\hbar(M_i))$ corresponding to the BV-operators Δ_i, $i=1,2$, are given by Lemma 2.

Proof. By Lemma 2, the homological vector fields Q_i regarded as infinitesimal shifts on the supermanifold $\mathbf{C}^{\infty}_{\hbar}(M_i)$ are given by

$$Q_i(f) = e^{-\frac{i}{\hbar}f}\Delta_i\big(e^{\frac{i}{\hbar}f}\big),$$

so that $f \mapsto f + \varepsilon Q_i(f)$. We need to show that $\widehat{\Phi}^!$ commutes with these shifts. Indeed, let $g \in \mathbf{C}^{\infty}_{\hbar}(M_2)$; apply the infinitesimal shift by Q_2 followed by $\widehat{\Phi}^!$. We obtain

$$\begin{aligned}
\widehat{\Phi}^!(g + \varepsilon Q_2(g)) &= \widehat{\Phi}^!\Big(g + \varepsilon e^{-\frac{i}{\hbar}g}\Delta_2\big(e^{\frac{i}{\hbar}g}\big)\Big) = \frac{\hbar}{i}\ln\widehat{\Phi}^*\exp\bigg(\frac{i}{\hbar}\Big(g + \varepsilon e^{-\frac{i}{\hbar}g}\Delta_2\big(e^{\frac{i}{\hbar}g}\big)\Big)\bigg)\\
&= \frac{\hbar}{i}\ln\widehat{\Phi}^*\bigg(e^{\frac{i}{\hbar}g}\Big(1 + \varepsilon\frac{i}{\hbar}\,e^{-\frac{i}{\hbar}g}\Delta_2\big(e^{\frac{i}{\hbar}g}\big)\Big)\bigg) = \frac{\hbar}{i}\ln\widehat{\Phi}^*\Big(e^{\frac{i}{\hbar}g} + \varepsilon\frac{i}{\hbar}\,\Delta_2\big(e^{\frac{i}{\hbar}g}\big)\Big)\\
&= \frac{\hbar}{i}\ln\Big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g} + \varepsilon\frac{i}{\hbar}\,\widehat{\Phi}^*\big(\Delta_2\big(e^{\frac{i}{\hbar}g}\big)\big)\Big) = \frac{\hbar}{i}\ln\Big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g} + \varepsilon\frac{i}{\hbar}\,\Delta_1\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big)\Big)\\
&= \frac{\hbar}{i}\ln\widehat{\Phi}^*e^{\frac{i}{\hbar}g} + \varepsilon\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big)^{-1}\Delta_1\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big) = \widehat{\Phi}^!(g) + \varepsilon\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big)^{-1}\Delta_1\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big).
\end{aligned}$$

Here we used the commutativity condition (5.12). Now apply first $\widehat{\Phi}^!$ and then the infinitesimal shift by Q_1. We have

$$\widehat{\Phi}^!(g) + \varepsilon Q_1\big(\widehat{\Phi}^!(g)\big) = \widehat{\Phi}^!(g) + \varepsilon e^{-\frac{i}{\hbar}\widehat{\Phi}^!(g)}\Delta_1\big(e^{\frac{i}{\hbar}\widehat{\Phi}^!(g)}\big);$$

note that

$$e^{\frac{i}{\hbar}\widehat{\Phi}^!(g)} = \widehat{\Phi}^*e^{\frac{i}{\hbar}g}.$$

Hence

$$\widehat{\Phi}^!(g) + \varepsilon Q_1\big(\widehat{\Phi}^!(g)\big) = \widehat{\Phi}^!(g) + \varepsilon\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big)^{-1}\Delta_1\big(\widehat{\Phi}^*e^{\frac{i}{\hbar}g}\big),$$

which is exactly as above. Thus, $\widehat{\Phi}^!$ intertwines Q_1 and Q_2 as claimed. $\square$

Remark 17. The definition of the map $\widehat{\Phi}^!$ by formulas (5.13) and (5.14) is motivated by the stationary phase method (*before* the limit $\hbar \to 0$ is taken). By Theorem 10, we have $\lim_{\hbar\to 0}\widehat{\Phi}^! = \Phi^*$, where Φ is the classical thick morphism corresponding to a quantum thick morphism $\widehat{\Phi}$. On the other hand, this construction is entirely algebraic and makes sense, together with an analog of Theorem 14, in the abstract setting as follows.

Let Δ be an odd formal $\hbar$-differential operator on a commutative unital superalgebra A that satisfies $\Delta^2 = 0$; we call it a *BV-operator*. Call an algebra A endowed with such a Δ a *BV-algebra*. (This terminology is not standard, but is convenient for our present purpose.) Every BV-operator generates an infinite sequence of brackets by (5.5), which defines an $S_{\infty,\hbar}$-structure on the algebra A. In fact, an $S_{\infty,\hbar}$-structure is completely determined by its 0- and 1-brackets, as all the higher brackets are inductively obtained as the discrepancies in the Leibniz identities. Since we can recover Δ as $\Delta(a) = \frac{\hbar}{i}\{a\}_{\Delta,\hbar} + \{\varnothing\}_{\Delta,\hbar}a$ and the Jacobi identities will give $\Delta^2 = 0$, the notions of a BV-algebra and $S_{\infty,\hbar}$-algebra coincide. Note also that since the parameter $\hbar$ plays a formal role here, we can set $-i\hbar \equiv 1$; then being a "formal $\hbar$-differential operator" becomes an empty condition and the brackets (5.5) turn back into the original operations introduced by Koszul. Let A_1 and A_2 be BV-algebras and let $\Phi\colon A_1 \to A_2$ be an even linear transformation such that $\Phi\circ\Delta_1 = \Delta_2\circ\Phi$ (Φ is not assumed to be a homomorphism with respect to the associative multiplication). Call such a Φ a *BV-morphism*. Define

$$\Phi_! := \ln_\hbar \circ\, \Phi \circ \exp_\hbar.$$

(There is an obvious functoriality relation $(\Phi_1\circ\Phi_2)_! = \Phi_{1!}\circ\Phi_{2!}$. If Φ is a homomorphism, then $\Phi_! = \Phi$.)

Theorem 15. *If Φ is a BV-morphism, then $\Phi_!$ is an L_∞-morphism of the $S_{\infty,\hbar}$-structures.*

Proof. The proof of Theorem 14 applies verbatim. $\square$

Corollary 5 (to Theorem 14)**.** *If $\widehat{\Phi}\colon M_1 \to_q M_2$ is a BV-morphism, then the classical pullback $\Phi^* = \lim_{\hbar\to 0}\widehat{\Phi}^!$ is an L_∞-morphism of the classical S_∞-structures.*

Proof. Indeed, $\widehat{\Phi}^!$ is an L_∞-morphism of the $S_{\infty,\hbar}$-structures. Passing to the limit $\hbar \to 0$ gives the claim. $\square$

In [44], we showed, for S_∞-manifolds, that if a thick morphism $\Phi\colon M_1 \to M_2$ is Poisson, i.e., the master Hamiltonians on M_1 and M_2 are Φ-related, then the pullback Φ^* is an L_∞-morphism of the homotopy Schouten brackets. We shall now relate this with Theorem 14.

Theorem 16. *Let M_1 and M_2 be BV-manifolds and let $\widehat{\Phi}\colon M_1 \to_q M_2$ be a quantum BV-morphism. Then its classical limit $\Phi\colon M_1 \to M_2$ is a Poisson morphism for the induced S_∞-structures.*

Proof. Let $H_i \in C^\infty(T^*M_i)$, $i = 1, 2$, be the master Hamiltonians for the S_∞-structures on M_1 and M_2 arising from the BV-operators Δ_1 and Δ_2. In other words, H_1 and H_2 are the principal symbols of Δ_1 and Δ_2. We need to show that H_1 and H_2 are Φ-related, i.e., $\pi_1^* H_1 = \pi_2^* H_2$ on the canonical relation $\Phi \subset M_2 \times (-M_1)$ (see [44]). This is a Hamilton–Jacobi equation

$$H_1\biggl(x, \frac{\partial S}{\partial x}\biggr) = H_2\biggl((-1)^{\widetilde{q}}\,\frac{\partial S}{\partial q}, q\biggr), \tag{5.15}$$

where $S(x, q)$ is the generating function of Φ. We are given that

$$\Delta_1 \circ \widehat{\Phi}^* = \widehat{\Phi}^* \circ \Delta_2.$$

In order to deduce (5.15) from this, write Δ_1 and Δ_2 as integral operators:

$$(\Delta_1 u)(x) = \int Dx'\, Đp'\, e^{\frac{i}{\hbar}(x-x')p'} H_{1,\hbar}(x, p')\, u(x')$$

and

$$(\Delta_2 w)(y) = \int Dy'\, Đq'\, e^{\frac{i}{\hbar}(y-y')q'} H_{2,\hbar}(y, q')\, w(y').$$

Here $H_{1,\hbar}$ and $H_{2,\hbar}$ are full symbols, which are coordinate-dependent objects. When $\hbar \to 0$, we get from them the principal symbols $H_1 = H_{1,0}$ and $H_2 = H_{2,0}$, which we need, and they are well-defined functions on T^*M_1 and T^*M_2. We have

$$(\Delta_1 \widehat{\Phi}^* w)(x) = \int Dx'\, Đp'\, e^{\frac{i}{\hbar}(x-x')p'} H_{1,\hbar}(x, p') \int Dy\, Đq\, e^{\frac{i}{\hbar}(S_\hbar(x',q)-yq)} w(y)$$

and

$$(\widehat{\Phi}^* \Delta_2 w)(x) = \int Dy\, Đq\, e^{\frac{i}{\hbar}(S_\hbar(x,q)-yq)} \int Dy'\, Đq'\, e^{\frac{i}{\hbar}(y-y')q'} H_{2,\hbar}(y, q')\, w(y'),$$

where $S_\hbar(x, q)$ is the quantum generating function for $\widehat{\Phi}$. Take $w = e^{\frac{i}{\hbar}yc}$ as a “test function” as in Example 6 and obtain, respectively,

$$(\Delta_1 \widehat{\Phi}^* w)(x) = \int Dx'\, Đp'\, Dy\, Đq\, e^{\frac{i}{\hbar}(S_\hbar(x',q)+(x-x')p'+y(c-q))} H_{1,\hbar}(x, p')$$

and

$$(\widehat{\Phi}^* \Delta_2 w)(x) = \int Dy\, Đq\, Dy'\, Đq'\, e^{\frac{i}{\hbar}(S_\hbar(x,q)+y(q'-q)+y'(c-q'))} H_{2,\hbar}(y, q').$$

In each case, the integral is simplified by the integration giving a delta function and the subsequent integration with the delta function. This finally gives

$$(\Delta_1\widehat{\Phi}^*w)(x) = \int Dx'\, Đp'\; e^{\frac{i}{\hbar}(S_\hbar(x',c)+(x-x')p')} H_{1,\hbar}(x,p')$$

and

$$(\widehat{\Phi}^*\Delta_2 w)(x) = \int Dy\, Đq\; e^{\frac{i}{\hbar}(S_\hbar(x,q)+y(c-q))} H_{2,\hbar}(y,c).$$

Now we apply the stationary phase method. The stationary points for the phases are specified, respectively, by the equations

$$x^a - x'^a = 0, \quad \frac{\partial S_\hbar}{\partial x^a}(x',c) - p'_a = 0 \qquad \text{and} \qquad c_i - q_i = 0, \quad \frac{\partial S_\hbar}{\partial q_i}(x,q) - (-1)^{\tilde{\imath}} y^i = 0,$$

and both Hessians are equal to 1. Altogether we obtain

$$(\Delta_1\widehat{\Phi}^*w)(x) = e^{\frac{i}{\hbar}S_\hbar(x,c)}\, H_{1,\hbar}\Big(x, \frac{\partial S_\hbar}{\partial x}(x,c)\Big)\big(1+O(\hbar)\big)$$

and

$$(\widehat{\Phi}^*\Delta_2 w)(x) = e^{\frac{i}{\hbar}S_\hbar(x,c)}\, H_{2,\hbar}\Big((-1)^{\tilde{q}}\frac{\partial S_\hbar}{\partial q}(x,c), c\Big)\big(1+O(\hbar)\big).$$

The phase factors coincide; so, by eliminating them and setting $\hbar \to 0$, we arrive at the equality

$$H_1\Big(x, \frac{\partial S_0}{\partial x}(x,c)\Big) = H_2\Big((-1)^{\tilde{q}}\frac{\partial S_0}{\partial q}(x,c), c\Big),$$

as desired because $S_0 = S$ is the generating function of Φ. $\square$

Theorem 16 is similar to Egorov's fundamental theorem about canonical transformations of pseudodifferential operators [8, 9] (see also [11]), which was one of the chief early sources for the theory of Fourier integral operators [18]. More precisely, in Egorov's theorem, Fourier integral operators are constructed that intertwine pseudodifferential operators whose principal symbols are related by a canonical transformation. The statement of our Theorem 16 is analogous to the converse Egorov theorem. An analog of the direct Egorov theorem would be the following statement that should also be true: every S_∞-structure, i.e., homotopy Schouten brackets for an arbitrary manifold, can be lifted to an $S_{\infty,\hbar}$-structure or equivalently to a BV-operator Δ, and every Poisson thick morphism between S_∞-manifolds can be lifted to a quantum BV-morphism, which intertwines Δ_1 and Δ_2.

CONCLUSIONS AND DISCUSSION

Let us summarize what we have achieved so far. We have introduced a new class of morphisms between smooth manifolds (or supermanifolds). They include smooth maps, but are not themselves maps in the ordinary sense, i.e., not maps of sets. In practice they are described by their "generating functions" $S(x_1, p_2)$ depending on position variables on the source manifold and momentum variables on the target manifold as arguments. The geometric objects underlying such morphisms (which we called thick or microformal) are canonical relations between the cotangent bundles of the source and target, of a particular type maximally close to those induced by ordinary maps of the base manifolds. Namely, these are relations that project without degeneration onto the source manifold and onto the fibers of the cotangent of the target; for the latter condition to make invariant sense, we are forced to consider our relations as formal. Hence the generating functions are formal power expansions in the cotangent directions. This explains the terminology "microformal morphisms" and "microformal geometry." Since generating functions that differ by a constant define the same canonical relation and we actually need the functions themselves, not up to constants, we may think that we work with "framed" relations (meaning a choice of additive constants).

The composition of thick morphisms between (super)manifolds is of course the standard composition of relations; however, the statement is that the resulting relation is of the same type and that the composition law is itself formal. The generating function of the composition of two thick morphisms is expressed as a formal power expansion in their generating functions. This composition law is local (depends only on the values of the generating functions and their derivatives of orders bounded from above in each term of the expansion). It is obtained by an iterative procedure. A similar iterative procedure defines the action of a thick morphism on smooth functions, i.e., the pullback. A distinctive feature of the pullback is that it is in general a nonlinear transformation.

This nonlinearity, first of all, forces us to distinguish between even functions and odd functions. There are two parallel constructions, of "even" and "odd" thick morphisms, corresponding to these two cases. Secondly, since the pullback of functions by a thick morphism of supermanifolds is in general nonlinear and in particular non-additive, it cannot be a ring homomorphism in the ordinary sense. This at the first glance contradicts the philosophy of "space–algebra duality," according to which to "spaces" there correspond algebras (interpreted as algebras of functions) and to maps of spaces there correspond algebra homomorphisms (with reversed direction). However, it turns out that the derivative of the pullback by a thick morphism, which is automatically a linear transformation, is the pullback in the ordinary sense (by some perturbed map between the source and target) and hence is an algebra homomorphism. This naturally suggests a "nonlinear generalization" of the notion of algebra homomorphisms and the corresponding generalization of the algebra/geometry duality. Such a generalization is yet to be explored. The author wishes to stress that his initial motivation was very concrete, namely, to obtain a method of construction of L_∞-morphisms for homotopy Poisson structures,[14] and that microformal geometry is indeed successful for that and other applications, such as those to vector bundles and Lie algebroids.

Still, since we have obtained two new (formal) categories, in the versions adapted to even functions and to odd functions, whose objects are supermanifolds, this inevitably leads to further questions. Such are, in particular,

(1) extending the functoriality from functions to other geometric objects such as, for example, differential forms;

(2) if the previous is successful, obtaining, further, an action of thick morphisms (possibly nonlinear) on various cohomology spaces or, for example, on the Fukaya categories of the cotangent bundles;

(3) exploring, by making use of these larger classes of morphisms, what, for example, group objects in the "thick" sense would be, and what could be obtained by gluing by thick diffeomorphisms.

There are also other specific questions which can be addressed in future studies. For instance, is it possible to obtain a more efficient description of the power expansions which specify the pullback and the composition (perhaps, by some graphic calculus)?

In microformal geometry, particularly in applications to homotopy Poisson structures, arises prominently the Hamilton–Jacobi equation: for example, in the form of the infinitesimal action on functions by thick diffeomorphisms [42],

$$f(x) \mapsto f(x) + \varepsilon H\left(x, \frac{\partial f}{\partial x}\right),$$

[14] We mean both homotopy Poisson and homotopy Schouten structures, i.e., the strongly homotopy versions of even and odd Poisson brackets.

also as an expression of the condition for a thick morphism between homotopy Poisson manifolds to be (homotopy) Poisson, and in the formula for a homological vector field on the space of functions [44]. Such prominence of the Hamilton–Jacobi equation in our constructions, together with its fundamental relation to the Schrödinger equation in quantum mechanics, has led us to building the quantum version of microformal geometry. In it, nonlinear pullbacks by "classical" thick morphisms are replaced by Fourier integral operators of some special kind (resembling the early version of such operators studied by Fock, Vishik and Eskin, Fedoryuk, and Egorov in the 1950s–1960s). The "classical" thick morphisms (in the bosonic case) are recovered from "quantum" ones in the limit $\hbar \to 0$. This may be seen in hindsight as an elucidation of the classical picture. Since the first motivation for microformal morphisms was related to homotopy Poisson structures and their L_∞-morphisms, it was natural to ask about a similar application of quantum thick morphisms. This has turned out to be indeed possible by replacing master Hamiltonians by Batalin–Vilkovisky type Δ-operators (cf. [19, 20]). We see here a fascinating interplay between homotopy algebras and some purely algebraic ideas on the one hand and very classical ideas from partial differential equations, pseudodifferential operators and Fourier integral operators on the other. Obviously, just as in the classical version, there are plenty of questions for further study.

Appendix A. A VERSION OF THE STATIONARY PHASE FORMULA

Here we recall a general type stationary phase formula and give its particular version adapted for application to quantum thick morphisms considered in Sections 4 and 5. We are basically following Fedoryuk's approach [11], with some modification and simplification (and extending it to the super case). For a general type formula, we consider an integral of the form

$$I_\phi(a) = \int\limits_{\mathbb{R}^{n|2m}} Dx\, e^{\frac{i}{\hbar}\phi(x)} a(x). \tag{A.1}$$

Here $\hbar$ is a formal parameter and both functions $\phi(x)$ (called "phase") and $a(x)$ are assumed to be formal power series in $\hbar$ over nonnegative powers. To simplify the notation, we do not explicitly indicate this dependence on $\hbar$. It is assumed that $a(x)$ is compactly supported and the phase $\phi(x)$ has one stationary point on the support of $a(x)$. (Obviously a more general case is reduced to this one by using partitions of unity.) Denote this point by x_0. There is an expansion

$$\phi(x) = \phi(x_0) + \frac{1}{2} d^2\phi(x_0)(x - x_0) + \phi^+(x; x_0), \tag{A.2}$$

where the function $\phi^+(x; x_0)$ has a zero of order 3 at $x = x_0$. Assume that the quadratic form $d^2\phi(x_0)$ is nondegenerate (that is why we need the dimension $n|2m$). We rewrite the integral as

$$I_\phi(a) = e^{\frac{i}{\hbar}\phi(x_0)} \int\limits_{\mathbb{R}^{n|2m}} Dx\, e^{\frac{i}{\hbar}\frac{1}{2} d^2\phi(x_0)(x-x_0)} \left(e^{\frac{i}{\hbar}\phi^+(x;x_0)} a(x)\right), \tag{A.3}$$

which, apart from the factor, has the general form of

$$\int Dx\, e^{\frac{i}{\hbar}\frac{1}{2} Q(x-x_0)}\, u(x),$$

where $Q(x - x_0)$ is a nondegenerate quadratic form and $u(x)$ some "test function." (We suppress the domains of integration when convenient.) Any such integral can be expressed as an application of a (formal) differential operator as follows. For an arbitrary function or distribution $f(x)$, the equality

$$\int Dx\, f(x_0 - x) u(x) = \widetilde{f}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) u(x)\Big|_{x=x_0} \tag{A.4}$$

holds, where $\widetilde{f}(p)$ denotes the $\hbar$-Fourier transform of $f(x)$. Indeed, $f(x-x')$ and $\widetilde{f}(p)$ are, respectively, the kernel and full symbol of a translationally invariant operator. In particular, for a Gaussian oscillating exponential $E(x)=e^{\frac{i}{\hbar}\frac{1}{2}Q(x)}$ on $\mathbb{R}^{n|2m}$, its $\hbar$-Fourier transform is

$$\widetilde{E}(p)=c_{n|2m,\hbar}\,\frac{e^{\frac{i\pi}{4}\operatorname{sgn}Q}}{\sqrt{|\operatorname{Ber}Q|}}\,e^{-\frac{i}{\hbar}\frac{1}{2}Q^{-1}(p)}, \tag{A.5}$$

where $c_{n|2m,\hbar}=(2\pi\hbar)^{\frac{n}{2}}(i\hbar)^{-m}$. Here we use Q both for the quadratic form $Q(x)=x^ax^bQ_{ba}$ and for its matrix Q_{ab}, and $\operatorname{sgn}Q$ is the signature (the difference of the numbers of positive and negative squares of the even variables in the canonical form). By $Q^{-1}(p)=Q^{ab}p_bp_a$ we denote the induced quadratic form on the momentum space, where $(Q^{ab})=(Q_{ab})^{-1}$. It is the super analog of the familiar formula and can be obtained by a manipulation with standard Gaussian integrals. Hence, for any function $u(x)$, we have

$$\int\limits_{\mathbb{R}^{n|2m}} Dx\, e^{\frac{i}{\hbar}\frac{1}{2}Q(x-x_0)}\,u(x)=c_{n|2m,\hbar}\,\frac{e^{\frac{i\pi}{4}\operatorname{sgn}Q}}{\sqrt{|\operatorname{Ber}Q|}}\,e^{-\frac{\hbar}{i}\frac{1}{2}Q^{-1}\left(\frac{\partial}{\partial x}\right)}u(x)\Big|_{x=x_0}. \tag{A.6}$$

By applying this to the integral (A.3), we arrive at the following statement.

Theorem 17 (a variant of [11, Theorem 2.3])**.** *For the integral $I_\phi(a)$, under the assumptions and in the notation above, there is a formula*

$$I_\phi(a)=c_{n|2m,\hbar}\,\frac{e^{\frac{i\pi}{4}\operatorname{sgn}d^2\phi(x_0)}}{\sqrt{|\operatorname{Ber}d^2\phi(x_0)|}}\,e^{\frac{i}{\hbar}\phi(x_0)}\left(e^{-\frac{\hbar}{i}\frac{1}{2}d^2\phi(x_0)^{-1}\left(\frac{\partial}{\partial x}\right)}\left(e^{\frac{i}{\hbar}\phi^+(x;x_0)}a(x)\right)\Big|_{x=x_0}\right), \tag{A.7}$$

where the expression in the big brackets is an expansion in nonnegative powers of $\hbar$ which equals $a(x_0)(1+O(\hbar))$ in the lowest order in $\hbar$.

Proof. All that is left to prove is the crucial observation that the result of the application of the operator $L=-\frac{\hbar}{i}\frac{1}{2}d^2\phi(x_0)^{-1}\left(\frac{\partial}{\partial x}\right)$ and its powers to the oscillating function

$$u(x)=e^{\frac{i}{\hbar}\phi^+(x;x_0)}a(x),$$

evaluated at $x=x_0$, does not contain negative powers of $\hbar$. This is because $\phi^+(x;x_0)$ has a zero of order 3 at $x=x_0$. Indeed, any derivative of order r of the function $u(x)$ is a sum of monomials of the form $a^{(k)}(b')^{k_1}(b'')^{k_2}\dots(b^{(r)})^{k_r}$, where $b(x):=\phi(x;x_0)$ and by $a^{(k)}$, b', b'', etc., we mean partial derivatives in x of orders k, 1, 2, etc. We have

$$k+k_1+2k_2+3k_3+\ldots+rk_r=r$$

and each such monomial carries a factor of $\hbar^{-1}$ to the power

$$k_1+k_2+k_3+\ldots+k_r,$$

which arises from differentiating the exponential $e^{\frac{i}{\hbar}b(x)}$. Consider $r=2s$ and let $k_1+k_2+k_3+\ldots+k_{2s}\geq s$. Then the monomial must contain the derivative b' or b''. (If it does not, i.e., $k_1=k_2=0$, then $k_3+\ldots+k_{2s}\geq s$ and the inequality $2s=k+k_1+2k_2+3k_3+\ldots+2sk_{2s}=k+3k_3+\ldots+2sk_{2s}\geq k+3(k_3+\ldots+k_{2s})\geq 3s$ holds, which is a contradiction.) Since $b'(x_0)=0$ and $b''(x_0)=0$, any partial derivative of $u(x)$ of order $2s$ at $x=x_0$ may contain $\hbar^{-1}$ only to the powers $<s$. Hence $L^su(x_0)$, for $s>0$, contains only positive powers of $\hbar$. Also $u(x_0)=a(x_0)$. So the expansion is as claimed. ☐

Now we consider a special case of integrals $I_\phi(a)$ where integration is over a $2n|2m$-dimensional space and the phase has the form

$$\phi(y,q)=S(q)-yq+\lambda g(y). \tag{A.8}$$

Here λ is a formal parameter. The functions $S(q)$ and $g(y)$ may depend on other variables not shown explicitly. In particular, they may be formal power series in $\hbar$. This type of phase function covers all the examples that we meet in Sections 4 and 5: quantum pullback, composition of quantum thick morphisms, transformation of coordinates and BV-morphisms. Let $S(q)$ be a formal power series in q_i,

$$S(q)=S^0+\varphi^i q_i+\frac{1}{2!}S^{ij}q_jq_i+\frac{1}{3!}S^{ijk}q_kq_jq_i+\dots \tag{A.9}$$

(while $g(y)$ be a smooth function). We write yq for y^iq_i and apply similar abbreviations. The integrals we are interested in have the form

$$I_\phi(a)=\int\limits_{\mathbb{R}^{2n|2m}} Dy\,\bar{D}q\; e^{\frac{i}{\hbar}\phi(y,q)}a(y,q), \tag{A.10}$$

where $\bar{D}q=(2\pi\hbar)^{-n}(i\hbar)^m Dq$. (Note that the factor is exactly $c^{-1}_{2n|2m,\hbar}$ in our notation.)

Lemma 4. *For the phase $\phi(y,q)$ given by* (A.8), *there is a unique stationary point (y_0,q^0), which is the (unique) solution of the equations*

$$y^i=(-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(q), \qquad q_i=\lambda\frac{\partial g}{\partial y^i}(y) \tag{A.11}$$

as perturbation series in λ,

$$y_0^i=\varphi^i+\lambda c^i_{(1)}+\frac{\lambda^2}{2!}c^i_{(2)}+\dots, \tag{A.12}$$

$$q_i^0=\lambda\frac{\partial g}{\partial y^i}(y_0)=\lambda\frac{\partial g}{\partial y^i}\Bigl(\varphi+\lambda c_{(1)}+\frac{\lambda^2}{2!}c_{(2)}+\dots\Bigr), \tag{A.13}$$

where the coefficients $c_{(k)}$ are homogeneous polynomials of degrees k in the derivatives of g of orders $\le k$ at $y=\varphi$,

$$c^i_{(1)}=S^{ij}\frac{\partial g}{\partial y^j}(\varphi), \qquad c^i_{(2)}=S^{ij}S^{kl}\frac{\partial g}{\partial y^l}(\varphi)\frac{\partial^2 g}{\partial y^k\,\partial y^j}(\varphi)+S^{ijk}\frac{\partial g}{\partial y^k}(\varphi)\frac{\partial g}{\partial y^j}(\varphi), \qquad \dots.$$

The stationary value is $\phi(y_0,q^0)=S^0+\lambda g(\varphi)+O(\lambda^2)$. The matrix of $d^2\phi(y_0,q^0)$ is

$$Q=\begin{pmatrix}\lambda\frac{\partial^2 g}{\partial y^i\,\partial y^j}(y_0) & -(-1)^{\tilde{\imath}}\delta_i{}^j\\ -\delta^i{}_j & \frac{\partial^2 S}{\partial q_i\,\partial q_j}(q^0)\end{pmatrix}. \tag{A.14}$$

Therefore, the stationary point (y_0,q^0) is nondegenerate; *for the Hessian, we have*

$$\bigl|\operatorname{Ber} d^2\phi(y_0,q^0)\bigr|=\operatorname{Ber}\Bigl(\delta_i{}^k-\lambda\frac{\partial^2 g}{\partial y^i\,\partial y^j}(y_0)\frac{\partial^2 S}{\partial q_j\,\partial q_k}(q^0)\Bigr)=1+O(\lambda). \tag{A.15}$$

Also, $\operatorname{sgn} d^2\phi(y_0,q^0)=0$.

Proof. Equations (A.11) are obtained by differentiating (A.8). They combine to give

$$y^i = (-1)^{\tilde{i}} \frac{\partial S}{\partial q_i}\left(\lambda \frac{\partial g}{\partial y}(y)\right),$$

solvable by iterations, giving a unique (y_0, q^0) as in (A.12), (A.13) with the claimed properties (cf. [44, Sect. 1]). The expression (A.14) for the Hesse matrix Q is obtained directly (for the relevant tensor notation and quadratic forms in the super case see, e.g., [43]). Its invertibility is clear from considering it in the zeroth order in λ. Equation (A.15) for $\operatorname{Ber} Q$ is obtained by multiplying the matrix Q by a matrix $J = \left(\begin{smallmatrix} 0 & -\delta^j{}_k \\ -\delta_j{}^k(-1)^k & 0 \end{smallmatrix}\right)$ with $\operatorname{Ber} J = \pm 1$, which gives $QJ = \left(\begin{smallmatrix} \delta_i{}^k & -\lambda g_{ik} \\ -s^{ik}(-1)^{\tilde{k}} & \delta^i{}_k \end{smallmatrix}\right)$, where $g_{ij} = \frac{\partial^2 g}{\partial y^i \partial y^j}(y_0)$ and $s^{ij} = \frac{\partial^2 S}{\partial q_i \partial q_j}(q^0)$, and by applying to the result the formula for the Berezinian of a block matrix (analogous to the well-known formula for the determinant). To see that the signature of $Q = d^2\phi(y_0, q^0)$ is zero, set $\lambda = 0$ and notice that by a linear change of the variables y^i the form can be brought to $Q = z^i q_i$. $\square$

Combining Lemma 4 with Theorem 17, we immediately obtain the desired statement.

Theorem 18. *For $\phi(y, q) = S(q) - yq + \lambda g(y)$ as in* (A.8), *we have*

$$\begin{aligned} I_\phi(a) &= \int\limits_{\mathbb{R}^{2n|2m}} Dy\, Đq\, e^{\frac{i}{\hbar}(S(q) - yq + \lambda g(y))} a(y, q) \\ &= e^{\frac{i}{\hbar}\phi(y_0, q^0)}\, b_0^{-\frac{1}{2}} \left(e^{-\frac{\hbar}{i}\frac{1}{2} L\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial q}\right)} \left(e^{\frac{i}{\hbar}\phi^+(y, q; y_0, q^0)} a(y, q)\right)\Big|_{y = y_0, q = q^0}\right). \end{aligned} \tag{A.16}$$

Here (y_0, q^0) is the stationary point given by (A.11)–(A.13). *The function $\phi^+(y, q; y_0, q^0)$ is as above. The matrix $L = Q^{-1}$ is the inverse for Q given by* (A.14), *so that*

$$L\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial q}\right) = L^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} + 2L^i{}_j \frac{\partial}{\partial q_j} \frac{\partial}{\partial y^i} + L_{ij} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_i},$$

and $b_0 = |\operatorname{Ber} Q|$ is given by (A.15). $\square$

Note that $\phi(y_0, q^0)$ in (A.16) has the form $\phi(y_0, q^0) = \phi_0 + O(\hbar)$, where ϕ_0 is the stationary phase value for $\phi_0(y, q)$ when $\hbar \to 0$. Also, $b_0 = b_{00} + O(\hbar)$, where b_{00} is invertible; hence the Hessian factor can be moved to the phase as a term of order ≥ 1 in $\hbar$. Finally, since the expression in the big brackets in (A.16) has the form $a_0 + O(\hbar)$, where a_0 is the "classical limit" of $a(y_0, q^0)$ when $\hbar \to 0$, we may say that

$$I_\phi(a) = e^{\frac{i}{\hbar}(\phi_0 + O(\hbar))}\big(a_0 + O(\hbar)\big). \tag{A.17}$$

In particular, if $a \equiv 1$, then $I_\phi(1) = e^{\frac{i}{\hbar}(\phi_0 + O(\hbar))}$. From the construction, we also see that both the phase and the amplitude of the integral $I_\phi(a)$ are formal power series in λ, which plays the role of a "coupling constant" (if we borrow the physicists' term). We do not use λ explicitly in the main text, speaking instead of expansions in the powers of the derivatives of the function g.

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