Hedging of the Barrier Put Option in a Diffusion (B, S) – Market in case of Dividends Payment on a Risk Active

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Abstract: Barrier European put option formed by additional clause putting in option contract with payment limitation for issuer and guaranteed income for holder of the security are researched when dividends on base risk active are paid. The equitable price, the optimal portfolio and a size of the capital answered the hedging strategy are founded for the options under consideration on diffusion (B, S)-financial market. Comparative price analysis for two option classes is carried out and specific properties of decision and decision under limiting are explored.

Keywords: Financial market, stochastic financial mathematics, option price, hedging strategy, barrier European put option, put option with guaranteed income for holder of the security, dividends.

1. INTRODUCTION

As of today the financial instruments of trading and risks hedging (Hull, 2013) on the derivatives market are presented by futures, forwards and options, particularly the exotic options (Rubinstein, 2013, Burenin, 2011ab, Shiryaev et al., 2006) The lasts are of interest for investor due to variety of the option’s payment liabilities (Shiryaev, 1999) and are the stochastic financial mathematics object (Melnikov et al., 2002). An European put option is a derivative (secondary) security, it is the contract giving option’s buyer (the holder) the right to sell stipulated underlying asset by a certain date for a certain price, and option’s seller must satisfy an agreement when exercising for an option premium (Hull, 2013).

The research is devoted to barrier European put option on stocks when dividends on base risk active are paid. The payoff function determined the payment size when the option under consideration exercising is

\[ f_T(S_T) = \min\{K_1 - S_T, K_2\} I[S_T > H] + (H - S_T) I[S_T \leq H]. \]

(1)

where \( S_T \) is risk asset’s spot price at expiration date \( T \), \( K_1 \) is exercise price or strike price, \( K_2 \) – contracted constant restricted payment of the option writer, on the one hand, and guaranteed income for option, \( H \) is barrier for price \( S_T \) (\( 0 \leq H < K_1 - K_2 \)). In accordance to (1) the exotic European put option payoff liability assumed as (2) is base for barrier option under study

\[ f_T^{\text{base}}(S_T) = \min\{K_1 - S_T, K_2\}. \]

(2)

and it goes on when intersection of barrier \( H \) by the spot price \( S_T \) top-down (in drop in prices phase). If at the moment \( T \) the market state such as \( S_T > H \) then the option holder gets the size \( f_T^{\text{base}}(S_T) \); in other cases (if \( S_T \leq H \) ) the option buyer earns rebate \( H - S_T \).

We denote the mathematical expectation by \( E\{\cdot\} \), the normal (Gaussian) density with the parameters \( a \) and \( b \) by \( N(a; b) \);

\[ \Phi(x) = \int \phi(y)dy, \quad \varphi(y) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] \] are Laplace distribution function and probability density function respectively.

2. STATEMENT OF THE PROBLEM

Let us consider complete, without arbitrage and risk-neutral financial market of two assets, notably: risk (stocks) and risk free (bank deposit) active. The stocks price evolution is given on stochastic basis \( (\Omega, F, \mathbb{F} = \{F_t\}_{t=0}^T, \mathbb{P}) \) (Shiryaev, 1999, Shiryaev et al., 2006). The current prices of the securities \( S_t \) on \( B_r, t \in [0, T] \), are specified by (3) and (4) respectively

\[ dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = S_0 \exp\left[\frac{\mu - (\sigma^2/2)t}{\sigma} + \sigma W_t\right]. \]

(3)

\[ dB_t = rB_t dt, \quad B_0 = B_0 \exp[rT], \]

(4)

where \( W_t \) is a standard Wiener process, \( S_0 > 0 \) is the stock initial cost, \( \mu \in \mathbb{R} = (-\infty, +\infty) \) is the percentage drift, \( \sigma > 0 \) is the percentage volatility in a geometric Brownian motion, \( B_0 > 0 \) is the risk free asset initial price, \( r > 0 \) is interest rate.

During time interval \( t \in [0, T] \) the investor forms self-financing portfolio \( \pi_t = (\beta_t, \gamma_t) \), where \( \mathbb{F}_t \)-measurable processes \( \beta_t \) and \( \gamma_t \) are parts of the risk free and risk assets.
at investment portfolio respectively, and this portfolio secures investor capital $X_\gamma = \beta B_t + \gamma S_t$. As in (Shiryayev, 1999, Shiryaev et al., 2006), for holding asset dividends are paid in accordance to process $D_t$ at the rate of $\delta_t S_t$, $\delta > 0$, that is $dD_t = \delta_t S_t dt$. Then capital change trajectory is described by equation $dx_t = \beta dB_t + \gamma dS_t + dD_t$. And as $dx_t = \beta dB_t + \gamma dS_t + \gamma B_t dB_t + \gamma S_t d\gamma_t - dD_t$ is a balance correlation replacing term $B_t dB_t + S_t d\gamma_t = 0$ for self financing portfolio in the standard problem (Burenin, 2011ab).

The problem involves the fact that to form the portfolio (hedging strategy) $\pi_t = (\beta_t, \gamma_t)$, the evolution of the capital $X_t$ has option price $P_t = X_0$ in accordance to the payoff function (1), as well as, the hedging strategy and corresponding capital, ensuring the fulfillment of payment liability $X_t = f_t(S_t)$.

3. PRELIMINARY RESULTS

All results below are obtained on the assumption of the sole risk-neutral measure existence. Relative to this measure the process of the risk asset capitalized price $S_t/B_t$ is martingale, and that condition guarantees the assigned problem solvability (Burenin, 2011ab, Shiryaev, 1999, Shiryaev et al., 2006, Melnikov et al., 2002).

Theorems 1, 2 are proved with a glance of the base financial relations (5)–(7) (Burenin, 2011ab, Shiryaev, 1999, Shiryaev et al., 2006, Melnikov et al., 2002)

$$ P_t = e^{-rT} E^\pi \{ f_t(S_T) \}, \quad (5) \quad \gamma_t = \frac{X_{(s)}(s)}{\pi_s}, \quad \beta_t = \frac{(X_t - \gamma_t S_t)}{B_t}, \quad (7) $$

where $E^\pi \{ \}$ – risk-neutral measure averaging.

Statement 1 (Shiryayev, 1999, Shiryaev et al., 2006). Let us that risk-neutral (martingale) measure $P^\ast = P^{\mu - r + \delta}$ is associated with source measure $P$ by transformation which looks like

$$ dP^{\mu - r + \delta} = Z^{\mu - r + \delta} dP, \quad (8) $$

where

$$ Z^{\mu - r + \delta} = \exp \left\{ - \frac{\mu - r + \delta}{\sigma} W_t - \frac{1}{2} \left( \frac{\mu - r + \delta}{\sigma} \right)^2 t \right\}. \quad (9) $$

Then stochastic properties of the process defined by equation

$$ dS_t = S_t (\mu dt + \sigma dW_t), \quad (10) $$

with regard to measure $P^{\mu - r + \delta}$ are coinciding with properties of the process $S(r, \delta)$ defined by equation

$$ dS_t(r, \delta) = S_t((r - \delta)dt + \sigma dW_t), \quad (11) $$

with respect to the measure $P$, where

$$ W^{r - \delta}_t = W_t + \frac{(\mu - r + \delta)}{\sigma} t \quad (12) $$

is Wiener process with respect to measure $P^{\mu - r + \delta} = P^\ast$.

4. MAIN RESULTS

Theorem 1. Let us consider the function below (13)–(16)

$$ y_1(T, S_0) = \left[ \ln(K_1/S_0) - (r - \delta - \sigma^2/2)T \right]/\sigma\sqrt{T}, \quad (13) $$

$$ y_2(T, S_0) = \left[ \ln((K_1 - K_2)/S_0) - (r - \delta - \sigma^2/2)T \right]/\sigma\sqrt{T}, \quad (14) $$

$$ y_3(T, S_0) = \left[ \ln(H/S_0) - (r - \delta - \sigma^2/2)T \right]/\sigma\sqrt{T}, \quad (15) $$

Then the value of the barrier European put option with payoff function (1) when dividends on risk asset are paid is defined as (17)

$$ P_t = K_t e^{-rT} [\Phi(y_1(T, S_0)) - \Phi(y_2(T, S_0))] - S_0 e^{-rT} \left[ \Phi(y_1(T, S_0)) - \Phi(y_2(T, S_0)) + \Phi(y_3(T, S_0)) + K_t e^{-rT} \right] \quad (17) $$

Proof. According to (1), (5) and using changes of variables $z = x/\sqrt{T}$, $y = z + ((\mu - r + \delta)/\sigma)/\sqrt{T}$ we obtain

$$ P_t = e^{-rT} E^\pi \left\{ \min \left\{ (K_1 - S_0)/, K_2, (S_T) > H \right\} + (H - S_T) \right\} \quad (18) $$

$$ \times I(S_T \leq H) = e^{-rT} \sqrt{2\pi} \int \exp \left\{ (y - 2/2) \right\} \min \left\{ (K_1 - S_0) exp((r - \delta - \sigma^2/2)T) + y\sigma/\sqrt{T} \right\} \left( H - S_t \right) exp((r - \delta - \sigma^2/2)T) + y\sigma/\sqrt{T} \right\} \leq H \right\} dy. $$

Obviously that (13)–(15) are roots of equations below

$$ S_0 \exp((r - \delta - \sigma^2/2)T) + \sigma(\sqrt{T}) = K_1, $$

$$ S_0 \exp((r - \delta - \sigma^2/2)T) + \sigma(\sqrt{T}) = K_2 $$

$$ S_0 \exp((r - \delta - \sigma^2/2)T) + \sigma(\sqrt{T}) = H. $$

So we get

$$ P_t = e^{-rT} \frac{e^{-rT}}{\sqrt{2\pi}} \int \exp \left\{ \frac{y^2}{2} \right\} \min \left\{ (K_1 - S_0) exp((r - \delta - \sigma^2/2)T) + y\sigma/\sqrt{T} \right\} dy = P_1 + P_2. $$
Summands $P_t^1$ and $P_t^2$ are defined by the formulas

$$
P_t^1 = K_t e^{-\delta t} \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] - S_t e^{-\delta t} \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] + K_te^{-\delta t} \Phi(y_2(T - t, S_0)) - \Phi(y_1(T - t, S_0)),
$$
(19)

$$
P_t^2 = He^{-\delta t} \Phi(y_1(T - t, S_0)) - S_t e^{-\delta t} \Phi(y_1(T - t, S_0)).
$$
(20)

Then, (17) holds if we substitute (19), (20) into (18).

**Theorem 2.** For the barrier put option with payoff function (1) the current values of the minimal hedging portfolio $\pi_t = (\beta_t, \gamma_t)$ and the accordance investment portfolio $X_t$ are described by (21)–(23)

$$
\begin{align*}
\gamma_t &= -e^{-\delta(t-t)} \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] + \Phi(y_1(T - t, S_0)) + K_te^{-\delta(t-t)} \frac{\sigma}{\sqrt{T - t}} \Phi(y_1(T - t, S_0)), \\
\beta_t &= (K_t / B_t) \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] + (K_t / B_t) \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] + H_Br \Phi(y_1(T - t, S_0)) - B_t \frac{\sigma}{\sqrt{T - t}} \Phi(y_1(T - t, S_0)), \\
X_t &= K_t e^{-\delta(t-t)} \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] - S_t e^{-\delta(t-t)} \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] + \Phi(y_1(T - t, S_0)) + K_te^{-\delta(t-t)} \Phi(y_2(T - t, S_0)) - \Phi(y_1(T - t, S_0)) + He^{-\delta(t-t)} \Phi(y_1(T - t, S_0)),
\end{align*}
$$
(21)

where $y_1(T - t, S_0), y_2(T - t, S_0), k = 1; 2; 3$, are defined by formulas (13)–(16) with substitutions $T \rightarrow (T - t), S_0 \rightarrow S_t$.

**Proof.** In accordance to (5), (6) formula (23) arises from (17) with replacements $T \rightarrow (T - t), S_0 \rightarrow S_t$.

According to (7), (23), we obtain

$$
\begin{align*}
\gamma_t &= -e^{-\delta(t-t)} \left[ \Phi(y_1(T - t, S_0)) - \Phi(y_2(T - t, S_0)) \right] + \Phi(y_1(T - t, S_0)) + K_te^{-\delta(t-t)} \left[ \frac{\partial \Phi(y_1(T - t, S_0))}{\partial s} \right]_{s=S_t} - S_t e^{-\delta(t-t)} \left[ \frac{\partial \Phi(y_1(T - t, S_0))}{\partial s} \right]_{s=S_t} + \Phi(y_1(T - t, S_0)) + K_te^{-\delta(t-t)} \left[ \frac{\partial \Phi(y_2(T - t, S_0))}{\partial s} \right]_{s=S_t} - \Phi(y_1(T - t, S_0)) + He^{-\delta(t-t)} \left[ \frac{\partial \Phi(y_1(T - t, S_0))}{\partial s} \right]_{s=S_t} + K_te^{-\delta(t-t)} \left[ \frac{\partial \Phi(y_2(T - t, S_0))}{\partial s} \right]_{s=S_t} - \Phi(y_1(T - t, S_0)),
\end{align*}
$$
(24)

In consideration of form of functions (13)–(16), we have the expressions

$$
\begin{align*}
&\frac{\partial \Phi(y_1(T - t, S_0))}{\partial s} \bigg|_{s=S_t} = -\frac{1}{\sqrt{2\pi}} \exp \left( \frac{(y_1(T - t, S_0))^2}{2} \right) \times \left[ \frac{1}{S_t \sigma \sqrt{T - t}} \right] = \frac{\phi(y_1(T - t, S_0))}{S_t \sigma \sqrt{T - t}},
\end{align*}
$$
(25)

$$
\begin{align*}
&\frac{\partial \Phi(y_1(T - t, S_0))}{\partial s} \bigg|_{s=S_t} = -\frac{1}{S_t \sigma \sqrt{T - t}} \phi(y_1(T - t, S_0)),
\end{align*}
$$
(26)

$$
\begin{align*}
&\frac{\partial \Phi(y_2(T - t, S_0))}{\partial s} \bigg|_{s=S_t} = -\frac{1}{S_t \sigma \sqrt{T - t}} \phi(y_2(T - t, S_0)),
\end{align*}
$$
(27)

$$
\begin{align*}
&\frac{\partial \Phi(y_1(T - t, S_0))}{\partial s} \bigg|_{s=S_t} = \frac{\phi(y_1(T - t, S_0))}{S_t \sigma \sqrt{T - t}},
\end{align*}
$$
(28)

The format of (25)–(28) follows from the definition of $P_t^S, P_t^K, P_t^H$ with (17).

**Statement 3.** The sensitivity coefficients (25)–(28) of barrier put option value with the option payoff function (1) satisfy the inequalities (29)

$$
P_t^S \wedge 0, \quad P_t^K \geq 0, \quad P_t^H \geq 0, \quad P_t^H \wedge 0.
$$
(29)

**Remark 1.** According to (1), (2)

$$
\lim_{H \rightarrow 0} f_t(S_t) = f_t^\text{base}(S_t) = \min \left( K_t - S_t \right),
$$
(30)

**Statement 4.** When $H \rightarrow 0$, we have $P_t \rightarrow P_t^\text{base}, X_t \rightarrow X_t^\text{base}, \gamma_t \rightarrow \gamma_t^\text{base}, \beta_t \rightarrow \beta_t^\text{base}$, where $P_t^\text{base}, X_t^\text{base}, \gamma_t^\text{base}, \beta_t^\text{base}$ are value, capital and investment portfolio of the exotic put option with payoff function (2) that defined in (Andrejko, 2010).
Remark 2. Statement 4 and (17), expressions for the exotic put option value from (Andreeva, 2010) make possible to compare options prices and obtain that $P_T \leq P_T^{base}$.

6. CONCLUSIONS

According to (29) analytically obtained properties $P_T^{K_1} > 0$, $P_T^{K_2} > 0$ are corroborated graphically (Fig. 1, 2) and can be interpreted with (1) as follows: strike price $K_1$ increment leads to probability that $K_1$ ranks over $S_T$ increase. Thus, payment size under exercising increases (if $S_T > H$) and derivative cost increases too. When $S_T > H$ the more size of the $K_2$ the more payment size for option emitter respectively. Option buyer risk decreases, and for less risk should pay more.

It is not succeed to establish analytically derivative value dependence on stock initial cost $S_0$ and on barrier $H$. However graphical solution (Fig. 3, 4) shows that $P_T^{S_0} < 0$, $P_T^{H} < 0$ and these properties can be explained as follows: at the average spot price increment is expected when value $S_0$ is more. Probability that $S_T$ ranks over exercise price $K_1$ increases. In this case, option buyer risk increases, and for this risk should pay less. The more size of the barrier $H$ the less probability that $S_T$ ranks over $H$. Consequently probability that $(H - S_T)$ will be paid increases. As $(H - S_T)$ is less than $f_T^{base}(S_T)$, barrier option price is decreasing function of barrier $H$.

REFERENCES


