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A NOTE ON ESSENTIALLY INDECOMPOSABLE *n*-SUMMABLE ABELIAN *p*-GROUPS

For each natural n we prove that there exists an unbounded n-summable abelian p-group which is essentially indecomposable. This example parallels a well-known result of this kind established for separable abelian p-groups.

Keywords: *summable groups, essentially indecomposable groups, admissible Ulm functions, direct sums of countable groups.*

0. Introduction and Fundamentals

Without any exceptions, the term "group" will mean an abelian p-group, where p is a prime fixed for the duration of the paper. Our terminology and notation will be based upon [1]. In particular, if G is a group and α is an arbitrary ordinal, then $p^{\alpha}G = \{x \in G:\ ht_G(x) \geq \alpha\}$, and we shall say G is *separable* if $p^{\omega}G = \{0\}$. Likewise, for every positive integer n, the symbol $G[p^n] = \{g \in G:\ p^n g = 0\}$ denotes the p^n-socle of G which can be viewed as a valuated group by consulting with [2]. About the notions of valuated p^n-socles, valuated groups and their closely related specifications, we refer the interested reader to [2] and [3].

The other specific concepts will be defined below explicitly as follows:

• Mimicking [2], a group G is said to *be n-summable* if $G[p^n]$ decomposes as (is isometric to) the valuated direct sum of a collection of countable valuated groups (each of which will also be a valuated p^n-socle).

Naturally, a group G is n-summable if $G[p^n]$ is n-summable as a valuated p^n-socle. Note that an n-summable group has to be summable (since a countable valuated vector space is necessarily free), and so $p^{\omega_1}G = \{0\}$ (see, e.g., Theorem 84.3 of [1]). In [3] was constructed for any natural n an n-summable group G which need *not* be n+1-summable such that $G / p^{\alpha}G$ is a direct sum of countable groups for all $\alpha < \omega_1$; thus this G is *not* a direct sum of countable groups.

• (Folklore) A group Z is said to be *essentially indecomposable* if whenever $Z \cong X \oplus Y$ for some groups X and Y, then either X or Y is bounded.

• Imitating [3], the function $f : \omega_1 \to C$ is called *n-realizable*, provided $f = f_V$ for some n-summable valuated p^n-socle V, where f_V designates the Ulm function of V. In particular, considering groups, $f = f_G$ for some n-summable group G, where $V = G[p^n]$.

• Imitating [3], the function $f : \omega_1 \to C$ is called *n-admissible*, provided it is n-closed and either uncountably unbounded or n-small and, in addition, for every pair of countable ordinals $\beta < \gamma$ with limit γ, the inequality $\sum_{[\gamma+n-1,\gamma+\omega)} f \le \left(\sum_{[\beta,\gamma)} f\right)^{\aleph_0}$ holds.

It can be proved that a function $f : \omega_1 \to C$ is n-admissible if, and only if, it is n-realizable (cf. [3]).

The motivation for writing this short article is to promote some new ideas concerning certain indecomposable properties of n-summable groups related to valuated groups and valuated p^n-socles (see, for more account, [4] and [5] too).

1. Examples and Assertions

If A is any separable group, B is a basic subgroup of A and $G = A/B[p^n]$, then the purity of B in A implies that there is an isomorphism

$$G[p^n] \cong \left(A[p^n]/B[p^n]\right) \oplus \left(B[p^{2n}]/B[p^n]\right).$$

Because B is ω-dense in A, it follows that the first term in this sum is $p^\omega G$. Considering multiplication by $p^n : B \to p^n B$, it follows that the second term is isometric to $p^n B[p^n]$ using the regular height function. It follows that $G[p^n]$ is n-summable and hence G is n-summable appealing to [2]. Note also that the isomorphism $G/p^\omega G \cong p^n A$ holds.

An example of an essentially indecomposable separable group Z can be constructed using Corollary 76.4 of [1]. So, we come to the following:

Example 1.1 *There is an n-summable group G that is essentially indecomposable.*

***Proof*:** If Z is a separable essentially indecomposable group and A is a separable group such that $p^n A \cong Z$, then let B be a basic subgroup of A and let $G = A/B[p^n]$, so that $G[p^n]$ is n-summable. If $G \cong X \oplus Y$, then

$$Z \cong p^n A \cong G/p^\omega G \cong \left(X/p^\omega X\right) \oplus \left(Y/p^\omega Y\right).$$

Therefore, either $\left(X/p^\omega X\right)$ or else $\left(Y/p^\omega Y\right)$ is bounded, so that either X or Y is bounded, which implies that G is also essentially indecomposable. ■

In other words, a group can have only inessential decompositions and still have a p^n-socle which splits into an infinite number of countable valuated summands.

In spite of the parallel between direct sums of countable groups and ω_1-bounded n-summable valuated p^n-socles, there are many n-summable groups that are not direct sums of countable groups. In fact, we have the following construction:

Example 1.2. *Any n-summable group G is a summand of a group with an admissible Ulm function that is not a direct sum of countable groups.*

***Proof*:** We can construct a direct sum of countable groups H which is large enough so that the Ulm function of $T = G \oplus H$ is admissible. This means that there is a direct sum of countable groups T' such that T and T' have the same Ulm functions. Since both $T[p^n]$ and $T'[p^n]$ are n-summable, they are isometric. On the other hand, T is not a direct sum of countable groups since this would imply that so is G – a contradiction. ■

Again, this shows that an n-summable group with the same p^n-socle as a direct sum of countable groups need not be a direct sum of countable groups. The next result characterizes the Ulm functions for which such a phenomenon can occur.

The following statement can also be deduced directly from results presented in [3], but we here give a more transparent proof, however.

Theorem 1.3. *Suppose $f : \omega_1 \to C$ is n-realizable. Then every n-summable group G with $f_G = f$ is a direct sum of countable groups if, and only if, $\sum_{[\omega+n-1,\omega_1)} f$ is countable.*

Proof: Suppose first that $\sum_{[\omega+n-1,\omega_1)} f$ is countable, and let H be $p^{\omega+n-1}$-high in G. Since G is n-summable, by Theorem 3.5 of [2], H must be a direct sum of countable groups. Since $r(G/H) = \sum_{[\omega+n-1,\omega_1)} f \le \aleph_0$, it follows from Wallace's theorem (see, for instance, Proposition 1.1 of [6]) that G is a direct sum of countable groups.

Conversely, suppose $\sum_{[\omega+n-1,\omega_1)} f$ is uncountable; our aim is then to produce an n-summable group G with $f_G = f$ which fails to be a direct sum of countable groups. If f is not admissible, then any n-summable group G with $f_G = f$ will fail to be a direct sum of countable groups, so we may assume that f is admissible. In particular, we can conclude that $\sum_{[\omega+n-1,\omega\cdot 2)} f$ is uncountable, so there is an integer $m \ge n-1$ such that $f(\omega+m)$ is uncountable. In addition, the admissibility of f implies that for every $\beta < \omega$, $\sum_{(\beta,\omega)} f$ is uncountable, so there is an unbounded subset $S \subseteq \omega$ such that for all $\beta \in S$, $f(\beta)$ is infinite.

We define

$$h(\beta) = \begin{cases} 1, & \text{if } \beta \in S; \\ \aleph_1, & \text{if } \beta = \omega+m; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathrm{supp}(h) = S \cup \{\omega+m\} \subseteq I_n$, it is clear that h is n-admissible, so there is an n-summable group H with $f_H = f$. Note that h is not admissible, so that H is not a direct sum of countable groups.

Since f is n-realizable, there is an n-summable group G' with $f_{G'} = f$. If $G = G' \oplus H$, then G is n-summable, and since it is easy to check that $f = f + h$, it follows that $f_G = f_{G'} + f_H = f + h = f$. On the other hand, since H fails to be a direct sum of countable groups, G is not a direct sum of countable groups, either. ■

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Для каждого натурального n мы доказываем, что существует неограниченная n-суммируемая абелева p-группа, которая существенно неразложима. Этот пример параллелен известному аналогичному результату, установленному для сепарабельных абелевых p-групп.

Ключевые слова: суммируемые группы, существенно неразложимые группы, допустимые функции Ульма, прямые суммы счетных групп.

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