Smooth Kernel Estimators of the Hazard Rate Function and its First and Second Derivatives

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***Abstract*—A class of nonparametric kernel estimators is suggested for an unknown hazard rate function and its derivatives. Both weak and mean square convergences of the proposed estimators to the unknown hazard function and its derivatives are proved. These estimators can be used for solving the problems of operational reliability of complex physical, technical, and program systems under uncertainty conditions.**

I. Introduction

At the present stage, successful development of our society calls for the development and introduction of complex systems and complexes in various spheres of man's activity. The design, manufacture, and operation of such systems call for maintenance of their reliability as one of the properties of systems to perform the required functions. Physicists face analogous problems when they estimate reliability of the created experimental models of chips, devices, setups, and their elements.

In the present paper, to evaluate reliability of system elements and to predict system failures, we suggest using the complete characteristic of reliability of nonrestorable elements, referred to as the hazard rate function, which has the form

$$\lambda(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{s(x)} \ , \qquad (1)$$

where $F(x)$ is the distribution function of failures of a nonrestorable element, $s(x) = 1 - F(x)$ is the reliability function, $f(x) = F'(x) = -s'(x)$ is the probability density function, and $x > 0$.

The hazard rate function $\lambda(x)$ characterizes local reliability of the element at each moment x and can be used to estimate the probability of failure for the given interval of time provided that before this moment there were no failures. It is important to note that to evaluate system reliability, it is convenient to use the well-known hazard rate functions of elements, because the formulas so derived are simple and convenient for engineering calculations [1].

Also, remark that a big application region of the hazard rate functions is connected with the queuing theory [2]. For example, there is a problem to estimate the amount of spare parts for machines working in different conditions. Namely, the queuing theory approach can be applied to find the required number of spare parts [3]. So, the reliability theory and, in particular, the hazard rate functions can be used in spare parts supply [4], [5].

The triplet $\{\lambda(x), \lambda'(x), \lambda''(x)\}$ is the complete characteristic of reliability of the nonrestorable element in comparison with $\lambda(x)$, where $\lambda'(x)$ is the derivative of the hazard rate function expressed by the formula

$$\lambda'(x) = \frac{f'(x)s(x) + f^2(x)}{s^2(x)} \ , \qquad (2)$$

which defines the rate of change of the hazard function at a point x, and the second derivative of $\lambda(x)$ has the form

$$\lambda''(x) = \frac{f''(x)s^2(x) + 3f(x)f'(x)s(x) + 2f^3(x)}{s^3(x)} \ , \qquad (3)$$

and it can be used to investigate the degree of smoothness of $\lambda(x)$.

II. Synthesis of Estimators

Let $X_1, \ldots, X_n$ be the moments of failures of the set of the checked elements. The work is aimed at the construction of estimators of the triplet $\{\lambda(x), \lambda'(x), \lambda''(x)\}$ from observations $\{X_i > 0,\ i = 1, \ldots, n\}$ under *a priori* nonparametric uncertainty given that only general information is available about both the distribution function $F(x)$ and reliability function $s(x)$. As nonparametric plug-in estimators of the triplet $\{\lambda(x), \lambda'(x), \lambda''(x)\}$, on account of formulas (1)–(3) and [6], we take

$$\lambda_n(x) = \frac{f_n(x)}{s_n(x)} \ , \qquad (4)$$

$$\lambda'_n(x) = \frac{f'_n(x)s_n(x) + f_n^2(x)}{s_n^2(x)} \ , \qquad (5)$$

$$\lambda''_n(x) = \frac{f''_n(x)s_n^2(x) + 3f_n(x)f'_n(x)s_n(x) + 2f_n^3(x)}{s_n^3(x)} \ , \qquad (6)$$

where

$$s_n(x) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right) ;$$

$K(u)$ is the thrice differentiated strictly monotonous decreasing function such that $K(-\infty) = 1$, $K(\infty) = 0$; the sequence $h_n \downarrow 0$; $\{X_i > 0,\ i = 1, \ldots, n\}$ is the sample of independent

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and identically distributed random variables from the general aggregate with the reliability function $s(x)$;

$$f_n(x) = \frac{-1}{nh_n} \sum_{i=1}^{n} K' \left(\frac{x - X_i}{h_n} \right) ;$$

$$f_n^{'}(x) = \frac{-1}{nh_n^2} \sum_{i=1}^{n} K'' \left(\frac{x - X_i}{h_n} \right) ;$$

$$f_n^{''}(x) = \frac{-1}{nh_n^3} \sum_{i=1}^{n} K''' \left(\frac{x - X_i}{h_n} \right) ;$$

$K'(u)$ is a symmetrical kernel-density, $u \in (-\infty, \infty)$. Here, $s_n(x)$ is a smooth empirical reliability function [7]–[9] ; $f_n(x) = f_n^{(0)}(x)$ is the nonparametric kernel estimator of the probability density function; $f_n'(x) = f_n^{(1)}(x)$ and $f_n''(x) = f_n^{(2)}(x)$ are estimators of the first and the second derivatives of the probability density function accordingly.

Let us designate by the symbol $\Longrightarrow$ the convergence in distribution and by $\mathcal{N}\{\mu, \sigma^2\}$ the one-dimensional random variable distributed normally with mean μ and variance σ^2, where $0 \le \sigma < \infty$, and by the symbols $\mathbf{E}$ and $\mathbf{D}$ denote the mathematical expectation and variance.

It is well known that the empirical reliability function $s_{n,emp}(x)$ is the simplest estimator of $s(x)$:

$$s_{n,emp}(x) = \frac{1}{n} \sum_{i=1}^{n} I\left(X_i > x\right),$$

where $I(A)$ is the indicator of an event A.

Present the properties of estimator $s_{n,emp}(x)$ [6].

1. Unbiasedness:

$$\mathbf{E}s_{n,emp}(x) = s(x).$$

2. Variance:

$$\mathbf{D}s_{n,emp}(x) = \frac{1}{n} s(x) \left(1 - s(x)\right).$$

3. According to the Central Limit Theorem

$$\sqrt{n}\left(s_{n,emp}(x) - s(x)\right) \Longrightarrow \mathcal{N}\left\{0, s(x)(1 - s(x))\right\}.$$

The estimator $s_{n,emp}(x)$ has two disadvantages:
1) $s_{n,emp}(x)$ is discontinuous at points $X_1, \ldots, X_n$,
2) $s_{n,emp}(x) = 0$ in a region

$$\Omega_\infty = (X_1 > t) \bigcap \cdots \bigcap (X_n > t).$$

It is clear that an estimator $\dfrac{f_n(x)}{s_{n,emp}(x)}$ [10] of a hazard rate function $\lambda(x)$ has also these disadvantages. For example, $\dfrac{f_n(x)}{s_{n,emp}(x)}$ is unusable in the region Ω_∞ [11] due to disadvantage 2.

To date, we know quite a lot of papers on nonparametric estimation of hazard rate functions (see, for example, [12]–[25]).

III. Asymptotic Normality

Construction of interval estimators with the given reliability for the hazard rate function and its derivative calls for the knowledge of limiting distribution of statistics $\lambda_n(x)$, $\lambda_n'(x)$, and $\lambda_n''(x)$. Now we specify the conditions when these statistics have asymptotically normal distributions.

Denote $L(r) = \int\limits_{-\infty}^{\infty} \left(K^{(r)}(u)\right)^2 du.$

Theorem 1. *Let the following conditions hold:*
1) $s(x) > 0$, $\lambda(x) > 0$;
for $r = 1, 2$:
2) *functions* $(r-1)f^{(r-2)}(x) + f^{(r-1)}(x)$ *are absolutely continuous on* R^1;
3) $K(x) = o(|x|^{-\alpha})$, $1 - K(-x) = o(|x|^{-\alpha})$ *as* $x \to \infty$, $\alpha > 0$;
$K^{(r)}(x) = o(|x|^{-\alpha - r})$ *as* $x \to +\infty$ *or* $x \to -\infty$;
4) *for all* $x \in R^1$ *the derivatives* $f^{(j+r)}(x)$, $j = 1, 2$, *are continuous;*
5) $\sup\limits_{x \in R^1} \left|f^{(j+r)}(x)\right| < \infty$, $j = 0, 1, 2$;
for $r = 0, 1, 2$:
6) $\lim\limits_{n \to \infty} \sqrt{nh_n^{1+2r}} \max\left(h_n^\alpha, h_n^2\right) = 0$,
$\lim\limits_{n \to \infty} \left(h_n + \dfrac{1}{nh_n^{1+2r}}\right) = 0$;
7) $L(1 + r) < \infty$.
Then, for $r = 0, 1, 2$ *and as* $n \to \infty$

$$\sqrt{nh_n^{1+2r}} \left[\lambda_n^{(r)}(x) - \lambda^{(r)}(x)\right] \Longrightarrow \mathcal{N}\left\{0, \frac{\lambda(x) L(1 + r)}{s(x)}\right\}. \tag{7}$$

Proof. The correctness of relation (7) for $r = 0$ follows from Theorem 4 of [11] with $\delta_n = 0$, $a_n = h_n$, $m = 2$, $l = \eta = \rho = \nu = 1$, for $r = 1$ according to [26], [27], whereas for $r = 2$ its correctness follows from Theorem 2.6.1 of the monograph [28].

Now consider for the triplet (1)–(3) other estimators:

$$\lambda_{n\mathcal{L}}(x) = \frac{f_n(x)}{s_{n\mathcal{L}}(x)}, \tag{8}$$

$$\lambda_{n\mathcal{L}}'(x) = \frac{f_n'(x)s_{n\mathcal{L}}(x) + f_n^2(x)}{s_{n\mathcal{L}}^2(x)}, \tag{9}$$

$$\lambda_{n\mathcal{L}}''(x) = = \frac{f_n''(x)s_{n\mathcal{L}}^2(x) + 3f_n(x)f_n'(x)s_{n\mathcal{L}}(x) + 2f_n^3(x)}{s_{n\mathcal{L}}^3(x)}, \tag{10}$$

where

$$s_{n\mathcal{L}}(x) = \frac{1}{n} \sum_{i=1}^{n} V_{\mathcal{L}} \left(\frac{x - X_i}{a_n} \right),$$

the Laplace kernel (see [7], [8])

$$V_{\mathcal{L}}(u) = \begin{cases} 1 - 0.5e^u, & -\infty < u < 0, \\ 0.5e^{-u}, & 0 \le u < \infty, \end{cases}$$

the sequence of positive numbers $a_n \downarrow 0$.

Define the conditions when estimators (8)–(10) have asymptotically normal distributions.

Theorem 2. *Let the following conditions hold:*

1) $s(x) > 0$, $\lambda(x) > 0$;

for $r = 1, 2$:

2) *functions* $(r-1)f^{(r-2)}(x) + f^{(r-1)}(x)$ *are absolutely continuous on* R^1;

3) *for all* $x \in R^1$ *the derivatives* $f^{(j+r)}(x)$, $j = 1, 2$, *are continuous;*

4) $\sup_{x\in R^1} \left| f^{(j+r)}(x) \right| < \infty$, $j = 0, 1, 2$;

for $r = 0, 1, 2$:

5) $\lim_{n\to\infty} \sqrt{nh_n^{1+2r}} \max\left(a_n, h_n^2\right) = 0$,

$\lim_{n\to\infty} \left(a_n + h_n + \frac{1}{nh_n^{1+2r}} \right) = 0$;

6) $L(1 + r) < \infty$.

Then, for $r = 0, 1, 2$ *and as* $n \to \infty$

$$\sqrt{nh_n^{1+2r}} \left[\lambda_{n\mathcal{L}}^{(r)}(x) - \lambda^{(r)}(x) \right] \Longrightarrow \mathcal{N}\left\{0, \frac{\lambda(x)L(1+r)}{s(x)}\right\}.$$

Proof. Here it was necessary to take into account that for the bias

$$b(s_{n\mathcal{L}}) = \mathbf{E}(s_{n\mathcal{L}}(x) - s(x))$$

holds the relation $|b(s_{n\mathcal{L}})| = O(a_n)$ (see [7], [8]).

IV. Interval Estimator

Based on (7) and the result of Theorem 2, the interval estimators can be constructed for the hazard rate function $\lambda(x)$. First, find the transformation of $\lambda_n(x)$, which has the limiting standard normal distribution. We formulate the following simple generalization of the result from the book [29] to obtain such transformation:

if a numerical sequence $q_n \uparrow \infty$, $\{t_n, n = 1, 2, \ldots\}$ *is a sequence of such statistics that*

$$\sqrt{q_n}[t_n - \theta] \Longrightarrow \mathcal{N}\left\{0, \sigma^2(\theta)\right\},$$

then

$$\sqrt{q_n}[g(t_n) - g(\theta)] \Longrightarrow \mathcal{N}\left\{0, [g'(\theta)\sigma(\theta)]^2\right\}, \qquad (11)$$

where g *is a function having the first derivative, and* $g'(\theta) \neq 0$.

According to [29], function g should be chosen as

$$g'(\theta)\sigma(\theta) = c, \qquad (12)$$

where c is independent of θ. In this case, the asymptotic variance of $g(t_n)$ does not depend on θ, but the function itself g can be found from the equation $g = \int \frac{cd\theta}{\sigma(\theta)}$. So, considering (7) for $r = 0$, we have

$$g(\lambda) = \int \frac{\sqrt{s(x)}d\lambda(x)}{\sqrt{\lambda(x)}\sqrt{L(1)}} = \int \frac{cd\lambda(x)}{\sqrt{\lambda(x)}}, \qquad (13)$$

and hence $g(x) = \sqrt{x}$.

Denote the bias of t_n by $b(t_n) = \mathbf{E}(t_n - t)$, where t_n is an estimator of t.

Theorem 3. *Under the conditions of Theorem 1 with* $r = 0$

$$\frac{2\sqrt{nh_n}\sqrt{s(x)}}{\sqrt{L(1)}} \left[\sqrt{\lambda_n(x)} - \sqrt{\lambda(x)} \right] \Longrightarrow \mathcal{N}\{0, 1\}. \qquad (14)$$

Proof. Assuming in the formula (11) $q_n = nh_n$, $t_n = \lambda_n$, $\theta = \lambda$, $g(\lambda) = \sqrt{\lambda}$, $g'(\lambda)\sigma(\lambda) = \frac{\sqrt{L(1)}}{2\sqrt{s(x)}}$, we obtain (14).

It follows from (20) that $|b(s_n(x))| = \frac{C}{\sqrt{n}}$, $0 < C < \infty$, and

$$\lim_{n\to\infty} |b(s_n(x))|\sqrt{nh_n} = C \lim_{n\to\infty} \sqrt{h_n} = 0.$$

Thus, in formula (14) validly replace $\sqrt{s(x)}$ to $\sqrt{s_n(x)}$, and under the conditions of Theorem 1 with $r = 0$

$$\frac{2\sqrt{nh_n}\sqrt{s_n(x)}}{\sqrt{L(1)}} \left[\sqrt{\lambda_n(x)} - \sqrt{\lambda(x)} \right] \Longrightarrow \mathcal{N}\{0, 1\}. \qquad (15)$$

Based on (15), the interval estimators can be constructed for the hazard rate function $\lambda(x)$. For example, the estimator of the $1 - \alpha$ confidence level has the following form:

$$\left(\sqrt{\lambda_n(x)} - \frac{\sqrt{s_n(x)L(1)}}{2\sqrt{nh_n}} u_{1-\alpha/2} \right)^2 < \lambda(x) <$$

$$< \left(\sqrt{\lambda_n(x)} + \frac{\sqrt{s_n(x)L(1)}}{2\sqrt{nh_n}} u_{1-\alpha/2} \right)^2, \qquad (16)$$

where $u_{1-\alpha/2}$ is the quantile of the level $1 - \alpha/2$ of the standard normal distribution.

Note that by Theorem 2 the inequality (16) holds and for the estimator $\lambda_{n\mathcal{L}}(x)$.

V. Mean-Square Error

One of the main accuracy characteristics of the estimator is its mean-square error (MSE). In the calculation of the MSEs of the plug-in estimators $\lambda_n(x)$, $\lambda_n'(x)$, and $\lambda_n''(x)$ some difficulties arise caused by their possible unboundness, for example, when the estimated denominators of $\lambda_n(x)$, $\lambda_n'(x)$, and $\lambda_n''(x)$ take zero values. In this connection, we now consider the piecewise-smoothed approximations of the estimators $\lambda_n^{(r)}(x)$, $r = 0, 1, 2$:

$$\Phi(\lambda_n^{(r)}(x), \delta_r) = \widetilde{\Phi}_r(x, \delta_r) = \frac{\lambda_n^{(r)}(x)}{(1 + \delta_r|\lambda_n^{(r)}(x)|^{\tau_r})^{\rho_r}}, \qquad (17)$$

where $\tau_r > 0$, $\rho_r > 0$, $\rho_r\tau_r \geq 1$, $\delta_r > 0$.

Now we introduce the following designations:

$t_n = (t_{1n}, \ldots, t_{sn})$ is the vector statistics with the components

$$t_{jn} = t_{jn}(x) = t_{jn}(x; X_1, \ldots, X_n), \quad j = 1, \ldots, s;$$

$$\|t_n\| = \sqrt{t_{1n}^2 + \cdots + t_{sn}^2}$$

is the Euclidean norm of the vector t_n;

$$\varphi = \varphi(x) = (\varphi_1(x), \ldots, \varphi_s(x))$$

is the bounded vector-function; R^n is the n-dimensional Euclidean space;

$$M_\nu\|t_n\| = \mathbf{E}\|t_n - \varphi\|^\nu$$

is the νth order moment of the deviation norm of the estimator $t_n(x)$ of the function $\varphi(x)$ at the point x.

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Definition. *The function* $H(z)$ *is said to enter the class* $\Gamma_{\nu,s}(\varphi)$ *if* $H(z) : R^s \to R^1$, *and for a certain fixed point* x *of the function* $\varphi = \varphi(x)$ *there exists* ε*-neighborhood* $\sigma = \{z : |z_i - \varphi_i| < \varepsilon,\ i = 1, \ldots, s\}$, *in which* $H(z)$ *and all its partial derivatives up to the* ν*th order inclusively are continuous and bounded.*

The function satisfying this definition is designated as $H(\cdot) \in \Gamma_{\nu,s}(\varphi)$.

Let N be the set of positive integers. Let us introduce for the triplet (τ, k, m) and $k, m \in N$ the set

$$T(m) = \Big\{(\tau, k) :\ \tau \geq \tau(m) = \frac{2k}{m-k-1} > 0,$$
$$m \geq m_0 = [3, k = 1; 2k, k \geq 2]\Big\}.$$

To calculate the mean-square errors of our estimators, we take the following result from [30], which below has been formulated as Theorem 4.

Let the function $H(\varphi) : R^s \to R^1$ and

$$\varphi = \varphi(x) = (\varphi_1(x), \ldots, \varphi_s(x))$$

be the bounded vector-function;

$$\nabla H(\varphi) = (H_1(\varphi), \ldots, H_s(\varphi)),$$
$$H_j(\varphi) = \partial H(z)/\partial z_j|_{z=\varphi}, \quad j = 1, \ldots, s\,;$$

$(d_n)_{n\geq 1}$ is a positive unboundedly increasing numerical sequence, that is $d_n \uparrow \infty$, as $n \to \infty$; T denotes transposition and C is nonnegative constant.

Theorem 4. *Let the following conditions hold:*

1) $H(z) \in \Gamma_{2,s}(\varphi)$;

2) *for a certain* $m \geq 3$, $m \in N$, $M_m\|t_n\| = O\left(d_n^{-m/2}\right)$;

3) $\delta = \delta_n = C d_n^{-1}$;

4) $H(\varphi) \neq 0$ *or* $\tau \in N$.

Then, for any $(\tau, k) \in T(m)$, *we have*

$$|\mathbf{E}[\widetilde{\Phi}(t_n, \delta_n) - H(\varphi)]^k - \mathbf{E}[\nabla H(\varphi)(t_n - \varphi)^{\mathrm{T}}]^k| =$$
$$= O\big(d_n^{-(k+1)/2}\big). \tag{18}$$

In accordance with condition 2, to use (18), we should know the order of convergence of the fourth moments of the estimators $s_n(x)$, $s_{n\mathcal{L}}(x)$, $f_n(x)$, $f_n'(x)$, and $f_n''(x)$. These results are presented below in the form of Lemmas 1– 3.

Lemma 1. *Let the following conditions hold:*

1) *the reliability function* $s(x)$ *is continuous at a point* x;

2) $K(x) = o(|x|^{-\alpha})$, $1 - K(-x) = o(|x|^{-\alpha})$ *as* $x \to \infty$, $\alpha > 0$;

3) *the sequence of numbers* $h_n = O\left(n^{-1/2\alpha}\right)$ *as* $n \to \infty$, $\alpha > 0$.

Then, as $n \to \infty$

$$M_4\|s_n(x)\| = \mathbf{E}\|s_n(x) - s(x)\|^4 =$$
$$= \mathbf{E}[s_n(x) - s(x)]^4 = O(n^{-2}). \tag{19}$$

Proof. On account of the inequality

$$\left(\sum_{i=1}^m |a_i|\right)^p \leq m^{p-1}\sum_{i=1}^m |a_i|^p, \quad p > 1,$$

for $p = 4$ and $m = 2$, we have

$$\mathbf{E}[s_n(x) - s(x)]^4 \leq 2^3[\mathbf{E}(s_n(x) - \mathbf{E}s_n(x))^4 + b^4(s_n(x))],$$

where $b(s_n(x)) = \mathsf{E}s_n(x) - s(x)$ is the bias of the estimator $s_n(x)$.

According to [31], $\mathbf{E}[s_n(x) - s(x)]^4 = O(n^{-2})$, and according to Lemma 2 from [11], the bias can be written as

$$b(s_n(x)) = O\left(h_n^\alpha\right) = O\left(\left[n^{-1/2\alpha}\right]^\alpha\right) = O\left(n^{-1/2}\right). \tag{20}$$

From the above the correctness of (19) immediately follows.

Let us evaluate the order of convergence for the fourth moments of the deviation norm of the estimators $f_n^{(r)}(x)$, $r = 0, 1, 2$, from the true functions $f^{(r)}(x)$. Denote

$$T = \int_{-\infty}^{\infty} u^2 K^{(1)}(u)du.$$

Lemma 2. *Let the following conditions hold:*

1) *the reliability function* $s(x)$ *is continuous at a point* x;

2) *the sequence of numbers* $a_n = O\left(n^{-1/2}\right)$ *as* $n \to \infty$.

Then, as $n \to \infty$

$$M_4\|s_{n\mathcal{L}}(x)\| = \mathbf{E}[s_{n\mathcal{L}}(x) - s(x)]^4 = O(n^{-2}). \tag{21}$$

Lemma 3. *Let the following conditions hold:*

1) $K'(u) < 0$, $K'(u) = K'(-u)$, $\sup_u K'(u) > -\infty$, $\int_{-\infty}^{\infty} K'(u)du = -1$, $T < \infty$;

2) $K(x) = o(|x|^{-\alpha})$, $1 - K(-x) = o(|x|^{-\alpha})$ *as* $x \to \infty$, $\alpha > 0$;

for $r = 0, 1, 2$:

3) *for all* $x \in R^1$ *the derivatives* $f^{(r)}(\cdot) \in \Gamma_{2+r,1}(x)$;

4) $\sup_{x\in R^1}\left|f^{(j+r)}(x)\right|$, $j = 0, 1, 2$;

5) $\lim_{n\to\infty}\left(h_n + \dfrac{1}{nh_n^{1+2r}}\right) = 0$;

for $r = 1, 2$ *the additional conditions hold:*

6) *functions* $(r-1)f^{(r-2)}(x) + f^{(r-1)}(x)$ *are absolutely continuous on* R^1;

7) $\int_{-\infty}^{\infty}\left|K^{(r)}(u)\right|du + (r-1)\int_{-\infty}^{\infty}\left|K^{(r-1)}(u)\right|du < \infty$;

8) $K'(x) = o(|x|^{-3})$ *as* $x \to +\infty$ *or* $x \to -\infty$;

for $r = 2$ *the condition holds*

9) $K'(x) = o(|x|^{-4})$ *as* $x \to +\infty$ *or* $x \to -\infty$.

Then, as $n \to \infty$

$$M_4\left\|f_n^{(r)}(x)\right\| = \mathbf{E}\left\|f_n^{(r)}(x) - f^{(r)}(x)\right\|^4 =$$
$$= \mathbf{E}\left[f_n^{(r)}(x) - f^{(r)}(x)\right]^4 = O\left(\left[h_n^4 + \frac{1}{nh_n^{1+2r}}\right]^2\right). \tag{22}$$

Proof. Reasoning similar to that used to prove (19), considering that in accordance with Lemma 2.1.3 from [28] $\mathbf{E}\left[f_n(x) - f(x)\right]^4 = O\left((nh_n)^2\right)$ and in accordance with Lemma 2.2.2 from [28] $b(f_n(x)) = O\left(h_n^2\right)$, we obtain (22) for $r = 0$. The case of $r = 1$ follows from Lemmas 2.1.7 and 2.2.6 of the monograph [28].

Let us now evaluate the MSEs of estimators $\lambda_n(x)$, $\lambda'_n(x)$, and $\lambda''_n(x)$.

Theorem 5. *Let for $r = 0, 1, 2$ the following conditions hold:*

1) $s(x) > 0$;
2) *the conditions of Lemmas 1 and 3 are fulfilled;*
3) $\delta_r = h_n^4 + \frac{1}{nh_n^{1+2r}}$;
4) $\lambda^{(r)}(x) \neq 0$;

Then, for any $(\tau_r, 2) \in T(m)$ as $r = 0, 1$, we have

$$\left| \mathbf{E}\left[\Phi(\lambda_n^{(r)}(x), \delta_r) - \lambda^{(r)}(x)\right]^2 - \frac{\lambda(x)L(1+r)}{nh_n^{1+2r}s(x)} - \right.$$
$$-\frac{T^2h_n^4}{4}\left(\frac{\left(f^{(2+r)}(x)\right)^2}{s^2(x)} + 4r\left[\lambda^2(x)\left(f^{(2)}(x)\right)^2 + \right.\right.$$
$$\left.\left.\left. +\frac{\lambda(x)}{s(x)}f^{(3)}(x)f^{(2)}(x)\right]\right)\right| =$$
$$= O\left(\left[h_n^4 + \frac{1}{nh_n^{1+2r}}\right]^{3/2}\right), \qquad (23)$$

and as $r = 2$

$$\left| \mathbf{E}\left[\Phi(\lambda_n^{(2)}(x), \delta_2) - \lambda^{(2)}(x)\right]^2 - \frac{\lambda(x)L(3)}{nh_n^5 s(x)} - \right.$$
$$-\frac{T^2h_n^4}{4}\left(\frac{\left(f^{(4)}(x)\right)^2}{s^2(x)} + 9\frac{\lambda^2(x)}{s^2(x)}\left(f^{(3)}(x)\right)^2 + \right.$$
$$+6\frac{\lambda(x)}{s^2(x)}\left(f^{(4)}(x)\right)^2 f^{(3)}(x) +$$
$$+6\frac{f^{(1)}(x)s(x) + 2f^2(x)}{s^4(x)}f^{(4)}(x)f^{(2)}(x) +$$
$$+9\left(\frac{f^{(1)}(x)s(x) + 2f^2(x)}{s^3(x)}\right)^2\left(f^{(2)}(x)\right)^2 +$$
$$\left.\left. +18\frac{\lambda(x)}{s(x)}\frac{f^{(1)}(x)s(x) + 2f^2(x)}{s^3(x)}f^{(3)}(x)f^{(2)}(x)\right]\right)\right| +$$
$$= O\left(\left[h_n^4 + \frac{1}{nh_n^5}\right]^{3/2}\right). \qquad (24)$$

Proof. Let us now prove the correctness of relation (23). For the case of $r = 0$ according to the notation of Theorem 4, we have

$$s = 2, z = (z_1, z_2), \quad H(z) = \frac{z_1}{z_2}, t_n = (t_{1n}, t_{2n}) = (f_n, s_n),$$

$$\varphi = (\varphi_1, \varphi_2), \varphi_1 = f(x), \varphi_2 = s(x), d_n = O\left(\frac{1}{h_n^4} + nh_n\right),$$

$k = 2$.

Let us take $m = m_0 = 4$ and prove that

$$M_4\|f_n, s_n\| = O(d_n^{-2}).$$

This immediately follows from (19), (22) (for $r = 0$), and the inequality

$$M_4\|f_n, s_n\| \le 2\left[M_4\|f_n\| + M_4\|s_n\|\right].$$

Since $s(x) > 0$, considering that for $z_2 > 0$ the function $\frac{z_1}{z_2} \in \Gamma_{2,2}(f, s)$, all the conditions necessary for the fulfillment of formula (18) to evaluate $\lambda_n^{(0)}(x) = \lambda_n(x)$ for $\tau > \tau_0 = 4$ are satisfied.

Analogously, for $r = 1$ we have

$$s = 3, \quad z = (z_1, z_2, z_3), \quad H(z) = \frac{z_3 z_2 + z_1^2}{z_2^2},$$

$$t_n = (t_{1n}, t_{2n}, t_{3n}) = (f_n, s_n, f'_n), \quad \varphi = (\varphi_1, \varphi_2, \varphi_3),$$

$$\varphi_1 = f(x), \quad \varphi_2 = s(x), \quad \varphi_3 = f'(x),$$

$$d_n = O\left(\frac{1}{h_n^4} + nh_n^3\right), \quad k = 2.$$

It is clear that

$$M_4\|f_n, s_n, f'_n\| \le 4\left[M_4\|f_n\| + M_4\|s_n\| + \left[M_4\|f'_n\|\right]\right].$$

So, the relation

$$M_4\|f_n, s_n, f'_n\| = O\left(\frac{1}{h_n^4} + nh_n^3\right)$$

follows from (19) and (22) for $r = 0, 1$.

Now, we prove the relation (24). Similarly, as in the proof of (23) according to the notation of Theorem 4, we have

$$s = 4, \quad z = (z_1, z_2, z_3, z_4), \quad H(z) = \frac{z_4 z_2^2 + 3z_1 z_2 z_3 + 2z_1^3}{z_2^3},$$

$$t_n = (t_{1n}, t_{2n}, t_{3n}, t_{4n}) = (f_n, s_n, f'_n, f''_n),$$

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4), \quad \varphi_1 = f(x), \quad \varphi_2 = s(x), \quad \varphi_3 = f'(x),$$

$$\varphi_4 = f''(x), \quad d_n = O\left(\frac{1}{h_n^4} + nh_n^5\right), \quad k = 2.$$

Thus,

$$M_4\|f_n, s_n, f'_n, f''_n\| \le$$
$$\le 8\left[M_4\|f_n\| + M_4\|s_n\| + M_4\|f'_n\| + M_4\|f''_n\|\right],$$

and

$$M_4\|f_n, s_n, f'_n, f''_n\| = O\left(\frac{1}{h_n^4} + nh_n^3\right)$$

follows from (19) and (22) for $r = 0, 1, 2$.

VI. Conclusion

Based on the above analysis of the properties of estimators (4)–(6) and (8)–(10), we conclude the following:

1. Asymptotic normality of the simple plug-in estimators (4) and (8) gives to the researcher the opportunity to construct for the function $\lambda(x)$ interval estimators with the given reliability level under uncertainty conditions.

2. Classic approaches do not allow one to evaluate the important accuracy characteristics of plug-in estimators (4)–(6) and (8)–(10), namely, their MSEs. In the present paper, the piecewise-smoothed approximations for plug-in estimators have been obtained, for which the main parts of their asymptotic MSEs have been found.

As a result of investigations, it was established that in comparison with the well-known parametric and discrete nonparametric algorithms, the advantage of the suggested estimators in calculations of the strength reliability is that they allow more reliable data to be obtained on the reliability of technical products and their residual lifetimes to be estimated. Thus, the researcher can obtain additional information based on the piecewise-smoothed approximations of the hazard rate function and its derivatives when calculating the probability of failure-free operation of a pipeline from the data of strength tests of steels (see [32], p. 163).

Also, the proposed algorithms and the results obtained can be used in solving the problem of increasing the reliability of various systems processing, transmitting and storing information. Here are some examples of such use:

— synthesis of better tests for new fault models;

— study of temporal models of the components of information systems;

— analysis and synthesis of controllers used in modern transport systems;

— construction of mixed diagnostic tests for hybrid intelligent training and testing system [33].

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