



Legendre transformations and Clairaut-type equations



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ABSTRACT

It is noted that the Legendre transformations in the standard formulation of quantum field theory have the form of functional Clairaut-type equations. It is shown that in presence of composite fields the Clairaut-type form holds after loop corrections are taken into account. A new solution to the functional Clairaut-type equation appearing in field theories with composite fields is found.

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1. Introduction

In quantum field theory the main object containing all possible information about a given dynamical system in quantum field theory is the generating functional of vertex functions or, in other words, the effective action. The usual way to introduce the effective action is by means of the Legendre transformation applied to the generating functional of connected Green functions. The relation between the effective action and the generating functional of connected Green functions has the form of functional Clairaut-type equation (see recent discussion in [1]). However, the perturbative (loop) expansion in the effective action does not preserve the mentioned Clairaut-type form of the Legendre transformation. This may explain why in the quantum field theory this specific feature of the Legendre transformation has been ignored.

In the present paper we are motivated by recent works [2, 3] devoted to the development of a new method to the functional renormalization group approach [4–6] and the study of the average effective action with composite fields. The approach to quantum field theory with composite fields has been developed by Cornwall, Jackiw and Tomboulis [7] in attempts to study physical phenomena (spontaneous symmetry breakdown, bound states, etc.) which cannot be easily considered in the perturbation (loop) expansion. Generalization of this method to gauge theories [8] de-

tected a special form of gauge dependence of the effective action with composite fields which in turn allowed to formulate the approach to functional renormalization group [2] being free of the gauge dependence problem inherent to the standard one [5,6]. Introduction of the effective action with composite fields requires to use the double Legendre transformations which as compared with the standard case cannot be presented in the form of functional Clairaut-type equation. Nevertheless, the perturbation series in the effective action with composite fields leads exactly to the functional Clairaut-type equation.

The paper is organized as follows. In Section 2 the relations existing between the effective actions without composite fields and with composite fields and functional Clairaut-type equations are derived. In Section 3 we study in detail solutions to the first-order partial Clairaut-type equations with a special form of the right-hand side and then generalize this result to the case of functional Clairaut-type equations. In Section 4 the way to find the one-loop correction to the effective action with composite fields by solving an appropriate functional Clairaut-type equation is shown. The remarkable result is that the effective action without composite fields does not offer such a solution. In Appendix A the simplest example of a set of matrices playing a very important role in solving the functional Clairaut-type equation which appears in field theories with composite fields is given.

2. Effective actions and Clairaut-type equations

Let us consider a field model which is described by a non-degenerate action, $S[\phi]$, of the scalar field $\phi = \phi(x)$. The generating

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functional of the Green functions, $Z[J]$, and the generating functional of the connected Green functions, $W[J]$, are defined in the standard way¹

$$Z[J] = \int \mathcal{D}\phi e^{i(S[\phi] + J\phi)} = e^{iW[J]}, \quad (2.1)$$

where $J = J(x)$ is usual source to ϕ and $J\phi = \int dx J(x)\phi(x)$. The effective action, $\Gamma = \Gamma[\Phi]$, is introduced by the Legendre transformation of $W[J]$

$$\Gamma[\Phi] = W[J] - J\Phi, \quad (2.2)$$

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x), \quad \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x). \quad (2.3)$$

Eliminating the source J from (2.2) one obtains the equation ($\Gamma = \Gamma[\Phi]$)

$$\Gamma - \frac{\delta \Gamma}{\delta \Phi} \Phi = W \left[-\frac{\delta \Gamma}{\delta \Phi} \right], \quad (2.4)$$

which has exactly the form of Clairaut-type equation. The practical application of Eq. (2.4) requires solving the equation (2.1) for $W[J]$. The standard way to find a solution to Eq. (2.4) with the functional W defined in Eq. (2.1) is related to using the perturbation theory. In the tree approximation, $\Gamma[\Phi] = \Gamma^{(0)}[\Phi]$, it follows $\Gamma^{(0)}[\Phi] = S[\Phi]$. In the one-loop approximation, $\Gamma[\Phi] = S[\Phi] + \Gamma^{(1)}[\Phi]$, one derives

$$\Gamma^{(1)}[\Phi] = \frac{i}{2} \text{Tr} \ln S''[\Phi], \quad (2.5)$$

where $S''[\Phi]$ means the functional matrix with elements

$$S''[\Phi](x, y) = \frac{\delta^2 S[\Phi]}{\delta \Phi(x) \delta \Phi(y)}, \quad (2.6)$$

and so on. Here and below we assume that the functional trace, Tr , can be defined in an appropriate way for functional matrices under consideration. In what follows we will use the following properties of functional trace

$$\text{Tr}(M + N) = \text{Tr} M + \text{Tr} N, \quad \text{Tr} MN = \text{Tr} NM, \quad (2.7)$$

for any two suitable functional matrices M and N as well as the relation between the functional determinant and the functional trace

$$\text{Det} M = \exp\{\text{Tr} \ln M\}. \quad (2.8)$$

In fact, specific features of Eq. (2.4) being the Clairaut-type equation disappear within the perturbation expansion. But it is not the case when one studies the effective action with composite fields [7]. Indeed, let us consider the effective action with composite fields. The starting point of such an approach is the generating functional of Green functions $Z[J, K]$,

$$Z[J, K] = \int \mathcal{D}\phi e^{i(S[\phi] + J\phi + KL(\phi))} = e^{iW[J, K]}, \quad (2.9)$$

where $W = W[J, K]$ is the generating functional of connected Green functions. In Eq. (2.9) $J = J(x)$ and $K = K(x, y)$ are sources to field $\phi = \phi(x)$ and composite field $L(\phi) = L(\phi)(x, y)$, respectively, and the notation $KL(\phi) = \int dx dy K(x, y)L(\phi)(x, y)$ is used. Let $L(\phi)(x, y)$ depend quadratically on the field ϕ

$$L(\phi)(x, y) = \frac{1}{2} \phi(x) \phi(y). \quad (2.10)$$

The effective action with composite field, $\Gamma = \Gamma[\Phi, F]$, is defined by using the double Legendre transformation [7]

$$\Gamma[\Phi, F] = W[J, K] - J\Phi - K(L(\Phi) + \frac{1}{2}F), \quad (2.11)$$

$$\begin{aligned} \frac{\delta W[J, K]}{\delta J(x)} &= \Phi(x), \\ \frac{\delta W[J, K]}{\delta K(x, y)} &= L(\Phi)(x, y) + \frac{1}{2}F(x, y), \end{aligned} \quad (2.12)$$

$$\begin{aligned} \frac{\delta \Gamma[\Phi, F]}{\delta \Phi(x)} &= -J(x) - \int dy K(x, y)\Phi(y), \\ \frac{\delta \Gamma[\Phi, F]}{\delta F(x, y)} &= -\frac{1}{2}K(x, y). \end{aligned} \quad (2.13)$$

Eliminating the sources J and K from the (2.11) one obtains the equation

$$\begin{aligned} \Gamma - \frac{\delta \Gamma}{\delta \Phi} \Phi - \frac{\delta \Gamma}{\delta F} F &= W \left[-\frac{\delta \Gamma}{\delta \Phi} + 2\frac{\delta \Gamma}{\delta F} \Phi, -2\frac{\delta \Gamma}{\delta F} \right] \\ &\quad - 2\frac{\delta \Gamma}{\delta F} L(\Phi). \end{aligned} \quad (2.14)$$

Since the right-hand side of Eq. (2.14) depends on the fields Φ not only through derivatives of functional $\Gamma = \Gamma[\Phi, F]$, the Eq. (2.14) does not belong to the Clairaut-type equation. But, in contrast with Eq. (2.5), the one-loop approximation for the effective action with composite field, $\Gamma^{(1)} = \Gamma^{(1)}[\Phi, F]$, by itself satisfies the equation

$$\Gamma^{(1)} - \frac{\delta \Gamma^{(1)}}{\delta F} F = \frac{i}{2} \text{Tr} \ln \left(S''[\Phi] - 2\frac{\delta \Gamma^{(1)}}{\delta F} \right), \quad (2.15)$$

being exactly the Clairaut-type with respect to field F wherein the variable Φ should be considered as parameter.

3. Remarks on the solutions of Clairaut-type equations

A Clairaut equation is a differential equation of the form

$$y - y'x = \psi(y'), \quad (3.1)$$

where $y = y(x)$, $y' = dy/dx$ and $\psi = \psi(z)$ is a real function of z . It is well-known that the general solution of the Clairaut equation is the family of straight line functions given by

$$y(x) = Cx + \psi(C), \quad (3.2)$$

where C is a real constant. The so-called singular solution is defined by the equation

$$\psi'(y') + x = 0, \quad (3.3)$$

if a solution to Eq. (3.3) can be present in the form $y' = \varphi(x)$ with a real function φ of x . Then the solution to Eq. (3.3) can be presented in the form

$$y(x) = y(\xi) + \int_{\xi}^x dx \varphi(x). \quad (3.4)$$

Now let us examine a first-order partial differential equation

$$y - y'_i x^i = \psi(y'), \quad (3.5)$$

which is also known as the Clairaut equation [10]. Here $y = y(x)$ is the real function of variables $x \in \mathbf{R}^n$, $x = \{x^1, x^2, \dots, x^n\}$, $\psi = \psi(z)$ is a real function of variables $z = \{z_1, z_2, \dots, z_n\}$ and the notation

$$y'_i = \partial_i y(x) \equiv \frac{\partial y(x)}{\partial x^i}, \quad (3.6)$$

¹ We use the DeWitt's condensed notations [9] and restrict ourselves to the case of a real scalar field for the sake of simplicity.

is used. In terms of new function $z_i = z_i(x) = y'_i(x)$ the equation (3.5) rewrites

$$y - z_i x^i = \psi(z). \quad (3.7)$$

Differentiation of Eq. (3.7) with respect to x^i leads to a system of differential equations

$$\frac{\partial z_j}{\partial x^i} \left(\frac{\partial \psi}{\partial z_j} + x^j \right) = 0, \quad i = 1, 2, \dots, n. \quad (3.8)$$

In the case of the Hessian matrix vanishing $H_{ij} = 0$ where

$$H_{ij} = \frac{\partial z_j}{\partial x^i} = \frac{\partial^2 y}{\partial x^i \partial x^j}, \quad (3.9)$$

$z_i = C_i = \text{const}$. Therefore the solution to Eq. (3.5) is the family of linear functions

$$y(x) = C_i x^i + \psi(C), \quad C = \{C_1, C_2, \dots, C_n\}. \quad (3.10)$$

If $\det H_{ij} \neq 0$ then the equations (3.8) are reduced to the following system

$$\frac{\partial \psi}{\partial z_j} + x^j = 0, \quad j = 1, 2, \dots, n. \quad (3.11)$$

If there are any real solutions to the equations (3.11)

$$z_j = \varphi_j(x), \quad j = 1, 2, \dots, n, \quad (3.12)$$

then the solution to Eq. (3.5) is reduced to the system of partial first-order differential equations

$$y'_j(x) = \varphi_j(x), \quad j = 1, 2, \dots, n, \quad (3.13)$$

resolved with respect to derivatives. If the conditions of integrability

$$\partial_i \varphi_j(x) = \partial_j \varphi_i(x) \quad (3.14)$$

are fulfilled then in any simply connected domain $G \subset \mathbb{R}^n$ the system (3.13) is solvable and the solution to this system can be presented in the form

$$y(x) = y(\xi) + \int_{\xi}^x dx^i \varphi_i(x), \quad (3.15)$$

where the integration is performed along any rectifiable curve in G having endpoints $\xi = \{\xi_1, \xi_2, \dots, \xi_n\}$ and $x = \{x_1, x_2, \dots, x_n\}$ (see, for example, [10]).

Now taking into account the structure of Eq. (2.15) we consider the Eq. (3.7) with the following choice of function ψ

$$\psi(z) = \alpha \ln \left(1 - \beta (z_i a^i) \right), \quad (3.16)$$

where α, β are real nonzero parameters and $a = \{a^1, a^2, \dots, a^n\}$ forms a n -tuple constant vector in a vector space V . If the Hessian matrix (3.9) $H_{ij} = \partial_i z_j = 0, \forall i, j = 1, \dots, n$ then $z_i = C_i = \text{const}$ and the solution to the Eqs. (3.5), (3.16) reads

$$y(x) = C_i x^i + \alpha \ln \left(1 - \beta C_i a^i \right). \quad (3.17)$$

In case when $\det H_{ij} \neq 0$ the equations (3.11) for $z_i = z_i(x)$ have the form

$$-\frac{\alpha \beta}{1 - \beta (z_i a^i)} a^j + x^j = 0, \quad j = 1, 2, \dots, n. \quad (3.18)$$

Specific feature of this system is that the required structure $z_i a^i$ and $z_i x^i$ as functions of $x = \{x_1, x_2, \dots, x_n\}$ can be found

algebraically. Indeed, let us introduce the constant vector $b = \{b_1, b_2, \dots, b_n\}$ in the dual vector space V^* so that $(b_i a^i) = 1$. Multiplying the Eq. (3.18) by b_j and summing the results one obtains

$$\begin{aligned} \left(1 - \beta (z_i a^i) \right)^{-1} &= \frac{1}{\alpha \beta} (b_i x^i), \\ \Rightarrow z_i a^i &= \beta^{-1} - \alpha (b_i x^i)^{-1}. \end{aligned} \quad (3.19)$$

In turn multiplying the Eq. (3.18) by z_j and summing the results we have

$$-\frac{\alpha \beta (z_j a^j)}{1 - \beta (z_i a^i)} + (z_j x^j) = 0, \quad \Rightarrow z_i x^i = \beta^{-1} (b_i x^i) - \alpha, \quad (3.20)$$

where the equation (3.19) is used. Finally the solution to Eqs. (3.5), (3.16) can be presented in the form

$$y(x) = \beta^{-1} (b_i x^i) - \alpha \ln (b_i x^i) - \alpha + \alpha \ln \alpha \beta. \quad (3.21)$$

The results obtained above for the partial differential equations can be immediately extended to the case of functional Clairaut-type equations. Let $\Gamma = \Gamma[F]$ be a functional of fields $F^m = F^m(x)$, $m = 1, 2, \dots, N$, which are real integrable functions of real variables $x \in \mathbb{R}^n$. We use the notion of *functional Clairaut-type equations* for the equations of the form

$$\Gamma - \frac{\delta \Gamma}{\delta F^m} F^m = \Psi \left[\frac{\delta \Gamma}{\delta F} \right], \quad (3.22)$$

where $\Psi = \Psi[Z]$ is a given real functional of real variables $Z_m = Z_m(x)$, $m = 1, 2, \dots, N$. In Eq. (3.22) the following notation

$$\frac{\delta \Gamma}{\delta F^m} F^m = \int d^n x \frac{\delta \Gamma}{\delta F^m(x)} F^m(x) \quad (3.23)$$

is used. The functional derivatives are defined by the rule

$$\frac{\delta F^m(x)}{\delta F^k(y)} = \delta_k^m \delta(x - y). \quad (3.24)$$

We restrict ourselves to the following functional Ψ

$$\Psi[Z] = \alpha \ln \left(1 - \beta (Z_m A^m) \right), \quad (3.25)$$

where $A^m = A^m(x)$, $m = 1, 2, \dots, N$ is the set of given integrable functions of real variables x and α, β are real nonzero parameters. Omitting the case of a linear functional Γ and repeating almost word for word the arguments made above we derive the non-trivial solution to the Eqs. (3.22), (3.25)

$$\Gamma[F] = \beta^{-1} (B_m F^m) - \alpha \ln (B_m F^m) - \alpha + \alpha \ln \alpha \beta, \quad (3.26)$$

where $B_m = B_m(x)$, $m = 1, 2, \dots, N$, are field variables satisfying the condition $B_m A^m = \int dx B_m(x) A^m(x) = 1$.

4. The effective action with composite fields

In this Section we generalize the results obtained in [7] to the case of a field model described by a set of scalar bosonic fields $\phi^A(x)$, $A = 1, \dots, N$, with a classical non-degenerate action $S[\phi]$. Let $L^i(\phi) = L^i(\phi)(x, y)$, $i = 1, 2, \dots, M$, be composite non-local fields,

$$L^i(\phi)(x, y) = \frac{1}{2} \mathcal{A}_{AB}^i \phi^A(x) \phi^B(y), \quad (4.1)$$

where $\mathcal{A}_{AB}^i = \mathcal{A}_{BA}^i$ are constants. The generating functional of Green functions, $Z[J, K]$, is given by the following path integral

$$Z[J, K] = \int \mathcal{D}\phi e^{i(S[\phi] + J_A \phi^A + K_i L^i(\phi))} = e^{iW[J, K]}, \quad (4.2)$$

where $K_i = K_i(x, y)$, $i = 1, 2, \dots, M$, are sources to composite fields $L^i(\phi)(x, y)$. Here the notations

$$J_A \phi^A = \int dx J_A(x) \phi^A(x), \quad (4.3)$$

$$K_i L^i(\phi) = \int dx dy K_i(x, y) L^i(\phi)(x, y), \quad (4.4)$$

are used. From Eq. (4.2) we can construct the following relations

$$\frac{1}{2} \mathcal{A}_{AB}^i \frac{\delta^2 Z[J, K]}{\delta J_A(x) \delta J_B(y)} = i \frac{\delta Z[J, K]}{\delta K_i(x, y)}, \quad (4.5)$$

or, in terms of the functional $W[J, K]$,

$$\begin{aligned} \frac{1}{2} \mathcal{A}_{AB}^i \left[-i \frac{\delta^2 W[J, K]}{\delta J_A(x) \delta J_B(y)} + \frac{\delta W[J, K]}{\delta J_A(x)} \frac{\delta W[J, K]}{\delta J_B(y)} \right] \\ = \frac{\delta W[J, K]}{\delta K_i(x, y)}. \end{aligned} \quad (4.6)$$

We define the average fields $\Phi^A(x)$ and composite fields $F^i(x, y)$ as follows

$$\frac{\delta W[J, K]}{\delta J_A(x)} = \Phi^A(x), \quad (4.7)$$

$$\frac{\delta W[J, K]}{\delta K_i(x, y)} = L^i(\Phi)(x, y) + \frac{1}{2} F^i(x, y). \quad (4.8)$$

The effective action with composite fields, $\Gamma = \Gamma[\Phi, F]$, is defined by using the double Legendre transformation of $W[J, K]$, (4.7) and (4.8),

$$\Gamma[\Phi, F] = W[J, K] - J_A \Phi^A - K_i (L^i(\Phi) + \frac{1}{2} F^i). \quad (4.9)$$

One can eliminate the sources from Eq. (4.9) using

$$\frac{\delta \Gamma[\Phi, F]}{\delta \Phi^A(x)} = -J_A(x) - \int dy K_i(x, y) \mathcal{A}_{AB}^i \Phi^B(y), \quad (4.10)$$

$$\frac{\delta \Gamma[\Phi, F]}{\delta F^i(x, y)} = -\frac{1}{2} K_i(x, y). \quad (4.11)$$

The relation (4.6) rewritten in terms of $\Gamma[\Phi, F]$ reads

$$F^j(x, y) - i(G^{-1})^{AB}(x, y) \mathcal{A}_{AB}^j = 0, \quad (4.12)$$

where (G^{-1}) is the matrix inverse to G ,

$$G = \{G_{AB}(x, y)\},$$

$$G_{AB}(x, y) = \Gamma_{AB}''[\Phi, F](x, y) - 2 \frac{\delta \Gamma[\Phi, F]}{\delta F^i(x, y)} \mathcal{A}_{AB}^i, \quad (4.13)$$

and we have used the notation

$$\Gamma_{AB}''[\Phi, F](x, y) = \frac{\delta^2 \Gamma[\Phi, F]}{\delta \Phi^A(x) \delta \Phi^B(y)}. \quad (4.14)$$

In the one-loop approximation, $\Gamma[\Phi, F] = S[\Phi] + \Gamma^{(1)}[\Phi, F]$, the equation for one-loop contribution, $\Gamma^{(1)}$, to the effective action can be found using procedure similar to [3]. It has the form

$$\Gamma^{(1)} - \frac{\delta \Gamma^{(1)}}{\delta F^i} F^i = \frac{i}{2} \text{Tr} \ln \left(S_{AB}''[\Phi] - 2 \frac{\delta \Gamma^{(1)}}{\delta F^i} \mathcal{A}_{AB}^i \right), \quad (4.15)$$

being the exact functional Clairaut-type equation.

According to the general scheme which is discussed in the previous section, to solve the Eq. (4.15) we introduce new functions $Z_i(x, y)$

$$\frac{\delta \Gamma[\Phi, F]}{\delta F^i(x, y)} = Z_i(x, y), \quad (4.16)$$

and substituting them to the Eq. (4.15) we obtain

$$\Gamma^{(1)} = Z_i F^i + \frac{i}{2} \text{Tr} \ln Q, \quad (4.17)$$

where the matrix $Q = \{Q_{AB}(x, y)\}$ is defined as

$$Q_{AB}(x, y) = S_{AB}''[\Phi](x, y) - 2 Z_i(x, y) \mathcal{A}_{AB}^i. \quad (4.18)$$

Then varying the functional (4.17) with respect to F^i we obtain

$$\delta \Gamma^{(1)} = \delta Z_i F^i + Z_i \delta F^i + \frac{i}{2} \text{Tr} Q^{-1} \delta Q, \quad (4.19)$$

where Q^{-1} is the inverse to Q

$$\int dz (Q^{-1})^{AC}(x, z) Q_{CB}(z, y) = \delta_B^A \delta(x - y). \quad (4.20)$$

Taking into account the explicit form of Q (4.18) and keeping in mind Eq. (4.16) from Eq. (4.19) it follows

$$\begin{aligned} \frac{\delta \Gamma^{(1)}[\Phi, F]}{\delta F^j(x, y)} &= Z_j(x, y) + \int dz dz' \frac{\delta Z_i(z, z')}{\delta F^j(x, y)} F^i(z, z') - \\ &\quad - i \int dz dz' (Q^{-1})^{AB}(z, z') \frac{\delta Z_i(z', z)}{\delta F^j(x, y)} \mathcal{A}_{BA}^i \\ &= Z_j(x, y). \end{aligned} \quad (4.21)$$

Thus the equation defining non-trivial functions $Z_i(x, y)$ reads

$$F^j(x, y) - i(Q^{-1})^{AB}(y, x) \mathcal{A}_{BA}^j = 0. \quad (4.22)$$

Note that in the approximation considered here this equation coincides with (4.12).

To solve the equation (4.22) we introduce a set of matrices $\mathcal{B}_j = \{\mathcal{B}_j^{AB}\}$ by the relations

$$\mathcal{A}_{AB}^j \mathcal{B}_j^{CD} = \frac{1}{2} (\delta_A^C \delta_B^D + \delta_A^D \delta_B^C). \quad (4.23)$$

Then we have

$$F^j(x, y) \mathcal{B}_j^{AB} - i(Q^{-1})^{BA}(y, x) = 0, \quad (4.24)$$

or, due to the symmetry property of $Q_{AB}(x, y) = Q_{BA}(y, x)$,

$$F^j(x, y) \mathcal{B}_j^{AB} - i(Q^{-1})^{AB}(x, y) = 0. \quad (4.25)$$

From Eqs. (4.22) and (4.25) one deduces the relations

$$\mathcal{B}_j^{AB} \mathcal{A}_{BA}^i = \delta_j^i \quad \text{or} \quad \text{tr} \mathcal{B}_j \mathcal{A}^i = \delta_j^i. \quad (4.26)$$

In turn Eqs. (4.23) and (4.26) lead to the restriction on parameters N and M

$$\frac{1}{2} N(N+1) = M. \quad (4.27)$$

The condition (4.27) has a simple sense: in a given theory with the set of N fields ϕ^A there exist exactly the $(1/2)N(N+1)$ independent combinations of $\phi^A \phi^B$.

The solution to the equation (4.25) has the form

$$Z_i(x, y) \mathcal{A}_{AB}^i = \frac{1}{2} S_{AB}''[\Phi](x, y) - \frac{i}{2} (F^j \mathcal{B}_j)_{AB}^{-1}(x, y), \quad (4.28)$$

where

$$\int dz (F^j \mathcal{B}_j)_{AC}^{-1}(x, z) (F^j(z, y) \mathcal{B}_j)^{CB} = \delta_B^A \delta(x - y). \quad (4.29)$$

We can express the $Z_i F^i$ as a functional of Φ, F multiplying Eq. (4.22) by Z_j and using (4.28) with the result

$$Z_j F^j = \frac{1}{2} \text{Tr} \left((F^j \mathcal{B}_j) S''[\Phi] \right) - \frac{i}{2} \delta(0) N. \quad (4.30)$$

Finally substituting (4.28) into (4.17) and using (4.30) we find the one-loop effective action, $\Gamma^{(1)}[\Phi, F]$, in the form

$$\Gamma^{(1)}[\Phi, F] = \frac{1}{2} \text{Tr} \left((F^j \mathcal{B}_j) S''[\Phi] \right) - \frac{i}{2} \text{Tr} \ln \left(i(F^j \mathcal{B}_j) \right) - \frac{i}{2} \delta(0) N. \quad (4.31)$$

The expression for $\Gamma^{(1)}$ (4.31) generalizes the known result of [7] in the cases $A = B = 1$ and $j = 1$ when $\mathcal{A}_{11}^1 = 1$ and $\mathcal{B}_1^{11} = 1$. This generalization involves the introduction of the set of matrices \mathcal{B}_j obeying the properties (4.23). In Appendix A we give a simple example of matrices \mathcal{A}^j and \mathcal{B}_j satisfying all required properties.

5. Discussions

In the present article we have studied relations existing between the Legendre transformations in quantum field theory and the functional differential equation for effective action which has the form of functional Clairaut-type equation. We have found that specific features of this equation do not hold within the perturbation theory in a quantum field theory without composite operators. But it is not the case within the approach to the quantum field theory based on composite fields when perturbation expansion of the effective action leads exactly to a functional Clairaut-type equation with a special type of the right-hand side. Partial first-order differential equations of Clairaut-type were our preliminary step in the study of solutions to the problem. It was shown that in case when the right-hand side of the equation has the form inspired by the real situation in quantum field theory with composite fields the solution to that functional Clairaut-type equation can be found with the help of algebraic manipulations only. In our knowledge the solution (3.21) to the equations (3.5) and (3.16) can be considered as a new result in the theory of partial first-order differential equations of Clairaut-type. This result has been easily extended (see (3.26)) to the case of functional Clairaut-type equation (3.22) with the special right-hand side (3.25). We have found an explicit solution to the functional Clairaut-type equation appearing in the quantum field theory with composite fields to define one loop contribution to the corresponding effective action (4.31).

We have studied the case with maximum number of composite fields, $M = \frac{1}{2}N(N+1)$, being quadratic in the given scalar fields ϕ^A , $A = 1, \dots, N$. In a similar manner one can consider the situation when the number of composite fields is less than the maximum one, $L^i(\phi)(x, y) = \frac{1}{2} \mathcal{A}_{ab}^i \phi^a(x) \phi^b(y)$, $i = 1, \dots, M < \frac{1}{2}N(N+1)$, $a = 1, \dots, n < N$. Now the matrix of second derivatives of the classical action, $S''_{AB}[\Phi]$, should be presented in the block form

$$S''_{AB}[\Phi](x, y) = \begin{pmatrix} S''_{ab} & S''_{a\beta} \\ S''_{\alpha b} & S''_{\alpha\beta} \end{pmatrix}, \quad (5.1)$$

where $a, b = 1, \dots, n$ and $\alpha, \beta = n+1, \dots, N$. The equation for the one-loop contribution, $\Gamma^{(1)}[\Phi, F]$, to effective action takes the form

$$\Gamma^{(1)} - \frac{\delta \Gamma^{(1)}}{\delta F^i} F^i = \frac{i}{2} \text{Tr} \ln \left(\tilde{S}''_{ab} - 2 \frac{\delta \Gamma^{(1)}}{\delta F^i} \mathcal{A}_{ab}^i \right) + \frac{i}{2} \text{Tr} \ln S''_{\alpha\beta}, \quad (5.2)$$

where

$$\tilde{S}''_{ab} = S''_{ab} - S''_{a\alpha} (S''^{-1})^{\alpha\beta} S''_{\beta b}. \quad (5.3)$$

The solution to the Eq. (5.2) reads

$$\Gamma^{(1)}[\Phi, F] = \frac{1}{2} \text{Tr} \left((F^j \mathcal{B}_j)^{ab} \tilde{S}''_{bc}[\Phi] \right) - \frac{i}{2} \text{Tr} \ln \left(i(F^j \mathcal{B}_j)^{ab} \right) + \frac{i}{2} \text{Tr} \ln S''_{\alpha\beta}[\Phi] - \frac{i}{2} \delta(0) n. \quad (5.4)$$

Here the matrixes \mathcal{B}_i^{ab} are introduced in the same way as in (4.23) but for the \mathcal{A}_{ab}^i ones.

Extension of the results obtained above to gauge theories can be easily performed in a way used in papers [2,3] on the basis of supermathematics [11,12]. We are going to present such kind of generalizations in our further studies.

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Appendix A. The simplest example of the matrices \mathcal{A}_{AB}^i and \mathcal{B}_i^{AB}

In solving the equation (4.22) the existence of matrices \mathcal{B}_i^{AB} satisfying the properties (4.23) and (4.26) plays a crucial role. To support this assumption we consider a simple example. Namely, let us consider the matrices \mathcal{A}_{AB}^i for the case of two fields, $N = 2$, thus $A, B = 1, 2$ in the Eq. (4.1). According to the Eq. (4.27) we have $M = 3$ and $i = 1, 2, 3$. The simplest choice of matrices $\mathcal{A}^i = \{\mathcal{A}_{AB}^i\}$ reads

$$\mathcal{A}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}^3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (A.1)$$

or in the condensed notation

$$\mathcal{A}^i = \begin{pmatrix} \delta_1^i & \frac{1}{2} \delta_3^i \\ \frac{1}{2} \delta_3^i & \delta_2^i \end{pmatrix}. \quad (A.2)$$

Then it is easy to construct the matrixes $\mathcal{B}_i = \{\mathcal{B}_i^{AB}\}$

$$\mathcal{B}_i = \begin{pmatrix} \delta_i^1 & \delta_i^3 \\ \delta_i^3 & \delta_i^2 \end{pmatrix}, \quad (A.3)$$

which satisfy the conditions (4.23) and (4.26).

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