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Anomaly-induced effective action and Chern–Simons modification of general relativity

Sebastião Mauro [a], Ilya L. Shapiro [b,a,c]

[a] Departamento de Física, ICE, Universidade Federal de Juiz de Fora, CEP: 36036-330, Juiz de Fora, MG, Brazil
[b] Département de Physique Théorique and Center for Astroparticle Physics, Université de Genève, 24 quai Ansermet, CH-1211 Genéve 4, Switzerland
[c] Tomsk State Pedagogical University and Tomsk State University, Tomsk, 634041, Russia

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ABSTRACT

Recently it was shown that the quantum vacuum effects of massless chiral fermion field in curved space–time leads to the parity-violating Pontryagin density term, which appears in the trace anomaly with imaginary coefficient. In the present work the anomaly-induced effective action with the parity-violating term is derived. The result is similar to the Chern–Simons modified general relativity, which was extensively studied in the last decade, but with the kinetic terms for the scalar different from those considered previously in the literature. The parity-breaking term makes no effect on the zero-order cosmology, but it is expected to be relevant in the black hole solutions and in the cosmological perturbations, especially gravitational waves.

1. Introduction

The derivation and properties of conformal (trace) anomaly are pretty well known (see, e.g., [1] and also [2,3] for the technical introduction related to the present work). At the one-loop level the anomaly is given by an algebraic sum of the contributions of massless conformal invariant fields of spins 0, 1/2, 1 in a curved space–time of an arbitrary background metric. Recently, it was confirmed that the quantum effects of chiral (L) fermion produce an imaginary contribution which violates parity [4]. As a result, the anomalous trace has the form

$$\langle T^\mu_\mu \rangle = -\beta_1 C^2 - \beta_2 E_4 - a' \Box R - \tilde{\beta} F^2_{\mu\nu} - \beta_4 P_4 \,. \tag{1}$$

Here we have included the external electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for generality, also

$$C^2 = C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta} \; = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + \frac{1}{3}\, R^2 \tag{2}$$

is the square of the Weyl tensor in four-dimensional space–time and

$$E_4 = \frac{1}{4}\, \varepsilon^{\mu\nu\alpha\beta}\, \varepsilon^{\rho\sigma\lambda\tau}\, R_{\mu\nu\rho\sigma}\, R_{\alpha\beta\lambda\tau} \; = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2 \tag{3}$$

is the integrand of the Gauss–Bonnet topological term.

The β-functions are given by algebraic sums of the contributions of N_s scalars, N_f Dirac fermions and N_v massless vector fields. The explicit form is well known,

$$\begin{aligned}
(4\pi)^2\, \beta_1 &= \frac{1}{120}\, N_s + \frac{1}{20}\, N_f + \frac{1}{10}\, N_v \,, \\
(4\pi)^2\, \beta_2 &= -\frac{1}{360}\, N_s - \frac{11}{360}\, N_f - \frac{31}{180}\, N_v \,, \\
(4\pi)^2\, \beta_3 &= \frac{1}{180}\, N_s + \frac{1}{30}\, N_f - \frac{1}{10}\, N_v \,.
\end{aligned} \tag{4}$$

One can assume that a' in (1) is equal to β_3, but there is ambiguity, as will be discussed below. $\tilde{\beta}$ is the usual β-function of QED or scalar QED etc, depending on the model.

Furthermore, there is a parity-violating Pontryagin density term $\beta_4 P_4$, where

$$P_4 = \frac{1}{2}\, \varepsilon^{\mu\nu\alpha\beta}\, R_{\mu\nu\rho\sigma}\, R_{\alpha\beta}{}^{\rho\sigma} \,. \tag{5}$$

By dimensional reasons the term with P_4 is possible, but for a long time it was believed that this term, in fact, does not show up. However, in a recent paper [4] this term was actually found with a purely imaginary coefficient $\beta_4 = i/(48 \cdot 16\pi^2)$, as a contribution of chiral (left) fermions. The chirality is important here, because the contribution of the right-hand fermions is going to cancel the one of the left-hand fermions, so taking them in a pair would kill

E-mail address: ilyashapiro2003@yahoo.com.br (I.L. Shapiro).

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the effect. Let us also note that much earlier, in [5], the possibility of such a term coming from integrating out antisymmetric tensor field has been considered, also some general considerations were presented even earlier in [6] and more recently in [7].

Some questions arise due to the result of [4] and its physical interpretation. First, does the parity-violating term in the anomaly mean that the dynamics of gravity is affected in a significant way? Second, in case of a positive answer to the last question, does it mean that the chiral fermions are disfavoured theoretically, since they produce imaginary component in the gravitational field equations? The last possibility was discussed in [4] as a theoretical argument in favor of massive neutrino. The third question is whether the parity-odd terms in the anomaly have some relation to the Chern–Simons modification of 4d-gravity suggested in [8,9]. The theories of this sort were extensively investigated in the last decade, as one can see from the review [10] and other works on the subject. This question looks really natural, because the Chern–Simons-gravity is based on the action which includes the P_4-term with an extra scalar factor inside the integral. Let us note that the relation between parity-odd terms and anomalies in $D = 4$ was discussed, i.e., in [11] in relation to gravitational anomalies, so the novelty of the term (5) concerns only the trace anomaly.

The purpose of the present work is to address the questions formulated above. In order to do so, we derive the effective action of gravity by integrating conformal anomaly, and show that the result is a new version of the Chern–Simons 4d-gravity with a special form of the kinetic term for the scalar and some extra higher-derivative terms which are typical for this action. From the technical side most of the consideration is pretty well known, but we present full details in order to make it readable for those who are not familiar with the subject. The paper is organized as follows. In Section 2 we review the well-known scheme of deriving anomaly-induced effective action, with an extra parity-odd term corresponding to Pontryagin density. The anomaly-induced action provides a specific form of the kinetic term for the auxiliary scalar in Chern–Simons modified gravity. For this reason, in the last subsection we present a short review of the previous version of kinetic terms, which are known in the literature. Section 3 includes a general, mainly qualitative, discussion of the physical interpretation of the new parity-violating term. Finally, in Section 4 we draw our conclusions and suggest possible perspectives of a further work on the subject.

2. Integration of anomaly with parity-violating term

The integration of conformal anomaly (1) in $d = 4$ means solving the equation similar to the one for the Polyakov action in $d = 2$,

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\,\frac{\delta\,\bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = -\,\langle T^\mu_\mu\rangle$$
$$= \frac{1}{(4\pi)^2}\left(\omega C^2 + bE_4 + c\Box R + \tilde{b}F^2_{\mu\nu} + \epsilon P_4\right). \qquad (6)$$

Here we introduced useful notations $(\omega,\, b,\, c,\, \tilde{b},\, \epsilon) = (4\pi)^2\left(\beta_1,\, \beta_2,\, a',\, \tilde{\beta},\, \beta_4\right)$. The coefficient ϵ derived in [4] is imaginary, but we will not pay attention to this until the solution is found. The first reason for this is that this is technically irrelevant, and also it is, in principle, possible to have a real coefficient of the same sort at the non-perturbative level.

2.1. Conformal properties of Pontryagin term and anomaly

The solution of Eq. (6) is technically is not very complicated [12] in the usual theory without Pontryagin term, and it remains equally simple when this term is present. In order to understand this, let us make an observation that this term is conformal invariant in $d = 4$, simply because one can recast (5) in the form when the Weyl tensor replaces the Riemann tensor,

$$P_4 = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,C_{\mu\nu\rho\sigma}\,C_{\alpha\beta}{}^{\rho\sigma}\,. \qquad (7)$$

The proof of this statement is well known (see [13] for further developments), but for the convenience of the reader we present a proof in Appendix A. One can easily see that the *r.h.s.* of Eq. (6) consists of the three different terms, which can be classified according to [14]. One can distinguish *(i)* conformally invariant part $\omega C^2 + \tilde{\beta}F^2_{\mu\nu} + \beta_4 P_4$; *(ii)* the topological term bE_4 and *(iii)* surface term $c\Box R$.

In fact, the last division is not unambiguous. For example, in $d = 4$ both P_4 and bE_4 can be presented as total derivatives, and the term P_4 is not only topological, but also conformal, according to Eq. (7). Hence, the Gauss–Bonnet invariant can be attributed to two groups of terms and the Pontryagin density even to all three groups *(i)*, *(ii)* and *(iii)*. In any case, as the reader will see shortly, the conformal invariance of P_4 makes the inclusion of this term into anomaly-induced action a very simple exercise. We shall present some details only to achieve a self-consistent exposition of the consideration.

The simplest part is the $\Box R$-term, which can be directly integrated by using the relation

$$-\,\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\sqrt{-g}\,R^2 \;=\; 12\Box R\,. \qquad (8)$$

It is easy to see that in this case the solution is a local functional, that gives rise to the well-known ambiguity in the coefficient a' of the $\Box R$-term, which was discussed in details in [15].

Now, let us concentrate on the non-local part of anomaly-induced action.[1] The solution of (6) can be presented in the simplest, non-covariant form, in the covariant non-local form and in the local covariant form with two auxiliary fields. Let us start from the simplest case. By introducing the conformal parametrization of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\,e^{2\sigma(x)} \qquad (9)$$

one can use an identity

$$-\,\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta\,A[g_{\mu\nu}]}{\delta\,g_{\mu\nu}} = -\,\frac{1}{\sqrt{-\bar{g}}}\;\frac{\delta\,A[\bar{g}_{\mu\nu}\,e^{2\sigma}]}{\delta\sigma}\Bigg|_{\bar{g}_{\mu\nu}\to g_{\mu\nu},\,\sigma\to 0}\,. \qquad (10)$$

Here and below the quantities with bars are constructed using the metric $\bar{g}_{\mu\nu}$, in particular

$$\bar{F}^2_{\mu\nu} = F_{\mu\nu}\,F_{\alpha\beta}\,\bar{g}^{\mu\alpha}\,\bar{g}^{\beta\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (11)$$

Furthermore, we will need the conformal transformation rules

$$\sqrt{-g}\,W_k = \sqrt{-\bar{g}}\bar{W}^2_k\,, \qquad \text{where} \qquad \left(W_k = C^2,\ P_4,\ F^2\right), \qquad (12)$$

and

$$\sqrt{-g}(E - \frac{2}{3}\Box R) = \sqrt{-\bar{g}}(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma)\,,$$
$$\sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4\,, \qquad (13)$$

where

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,R_{;\mu}\,\nabla^\mu \qquad (14)$$

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[1] The non-localities due to anomaly was first discussed in [16].

is covariant, self-adjoint, fourth-derivative, conformal operator [17].

After we use the transformation rules (13) and (12), Eq. (6) becomes very simple and the solution for the effective action can be found in the form

$$\bar{\Gamma}_{ind} = \frac{1}{(4\pi)^2}\int d^4x\sqrt{-\bar{g}}\,\Big\{\omega\sigma\bar{C}^2 + \tilde{\beta}\sigma\bar{F}^2_{\mu\nu} + \epsilon\bar{P}_4 + b\sigma\big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}\big) + 2b\sigma\,\bar{\Delta}_4\sigma\Big\} - \frac{1}{12}\Big(c + \frac{2b}{3}\Big)\frac{1}{(4\pi)^2}\int d^4x\sqrt{-g}\,R^2 + S_c[\bar{g}_{\mu\nu},\,A_\mu], \qquad (15)$$

where $S_c[\bar{g}_{\mu\nu},\,A_\mu] = S_c[g_{\mu\nu},\,A_\mu]$ is an unknown conformal invariant functional of the metric and A_μ. This functional is an integration constant for Eq. (6) and hence it cannot be uniquely defined in the present framework. Let us note that in some cases this functional is irrelevant. An example is cosmological solution without background electromagnetic field. In this case the metric is conformally trivial and $S_c[g_{\mu\nu}]$ becomes an irrelevant constant.

Even in cases of non-cosmological metrics the functional $S_c[g_{\mu\nu}]$ does not prove to be very significant, because the rest of the effective action (15) contains all information about the UV behavior of the theory. In the massless case, with a usual duality between UV and IR regimes, this means that $S_c[g_{\mu\nu}]$ may have only sub-leading contributions. These arguments are confirmed by successful applications to black holes [18,19] and gravitational waves [20,21].

The solution (15) is non-covariant, because it is not expressed in terms of the physical metric $g_{\mu\nu}$. In order to obtain the non-local covariant solution of Eq. (6), one has to introduce the Green function for the Paneitz operator,

$$\big(\sqrt{-g}\Delta_4\big)_x\,G(x,y) = \delta(x,y) \quad \text{and notation} \quad \int_x = \int d^4x\sqrt{-g(x)}\,. \qquad (16)$$

Using (10) it is easy to check that for any conformal functional $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$,

$$2g_{\mu\nu}(y)\,\frac{\delta}{\delta g_{\mu\nu}(y)}\int_x A\cdot\big(E - \frac{2}{3}\Box R\big) = \frac{\delta}{\delta\sigma(y)}\int_x A\cdot\big(E - \frac{2}{3}\Box R\big)\Big|_{\sigma\to 0,\ \bar{g}_{\mu\nu}\to g_{\mu\nu}} = 4\sqrt{-\bar{g}}\bar{\Delta}_4\,A = 4\sqrt{-g}\Delta_4\,A\,. \qquad (17)$$

By means of the last relation it is easy to solve both remaining parts (remember that the local part we already have from Eq. (8)) of induced effective action, and we arrive at

$$\bar{\Gamma}_{ind} = \Gamma_\omega + \Gamma_b + \Gamma_c\,, \qquad (18)$$

where

$$\Gamma_\omega = \frac{1}{4}\int_x\int_y \big(\omega C^2 + \tilde{b}F^2_{\mu\nu} + \epsilon P_4\big)_x\,G(x,y)\,\big(E - \frac{2}{3}\Box R\big)_y\,, \qquad (19)$$

$$\Gamma_b = \frac{b}{8}\int_x\int_y \big(E - \frac{2}{3}\Box R\big)_x\,G(x,y)\,\big(E - \frac{2}{3}\Box R\big)_y \qquad (20)$$

and

$$\Gamma_c = -\frac{c + \frac{2}{3}b}{12(4\pi)^2}\int_x R^2(x)\,. \qquad (21)$$

One has to note that the Pontryagin density shows up only in the first nonlocal term (19), but in what follows we shall see that the second term (20) is still relevant for constructing the kinetic term of the Chern–Simons modification of gravity. At the same time, the local term (21) will remain separated from others.

2.2. Anomaly-induced action and kinetic term for Chern–Simons gravity

As a next step, the nonlocal expressions for the anomaly-induced effective action can be presented in a local form by introducing two auxiliary scalar fields φ and ψ [22]. An equivalent two-scalar representation was suggested in [23], while the simpler one-scalar form was known from much earlier [12]. Since the details of the procedure were described also in [2,3] and do not change essentially due to the term P_4, let us present only the final result

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2 + \int_x \Big\{\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi + \varphi\Big[\frac{\sqrt{-b}}{8\pi}\big(E - \frac{2}{3}\Box R\big) - \frac{1}{8\pi\sqrt{-b}}\Big(\omega C^2 + \tilde{b}F^2_{\mu\nu} + \epsilon P_4\Big)\Big] + \frac{1}{8\pi\sqrt{-b}}\,\psi\,\Big(\omega C^2 + \tilde{b}F^2_{\mu\nu} + \epsilon P_4\Big)\Big\}. \qquad (22)$$

At the classical level the local covariant form (22) is equivalent to the non-local covariant form (18). The definition of the boundary conditions for the Green functions $G(x,y)$ are equivalent to the same boundary conditions for the auxiliary scalars φ and ψ. For the discussion of the importance to have two fields let us address the reader to [22,23,2].

The action (22) represent a final product of our integration of conformal anomaly. However, in order to make the consideration leading to the version of Chern–Simons gravity [8] more explicit, let us make a change of variables similar to one of [23]. Let us introduce two new scalars,

$$\chi = \frac{\psi - \varphi}{\sqrt{2}}\,, \qquad \xi = \frac{\psi + \varphi}{\sqrt{2}}\,, \qquad (23)$$

such that

$$\varphi = \frac{\xi - \chi}{\sqrt{2}}\,, \qquad \psi = \frac{\chi + \xi}{\sqrt{2}}\,. \qquad (24)$$

Then the total gravitational action, including the classical part and the anomaly-induced action (22) can be cast into the form

$$\Gamma_{grav} = S_{EH} + S_{HD} + \Gamma_{ind} = S_{EH}[g_{\mu\nu}] + S_{HD}[g_{\mu\nu}] + S_c[g_{\mu\nu}] + \int_x \Big\{\xi\Delta_4\chi \,+\, k_1\big(E - \frac{2}{3}\Box R\big)(\xi - \chi) + k_2\,\chi\,\Big(\omega C^2 + \tilde{b}F^2_{\mu\nu} + \epsilon P_4\Big) + k_3 R^2\Big\}. \qquad (25)$$

where

$$k_1 = \frac{1}{8\pi}\sqrt{-\frac{b}{2}}\,, \qquad k_2 = \frac{1}{8\pi\sqrt{-2b}}\,, \qquad k_3 = -\frac{2b+3c}{36\,(4\pi)^2}\,, \qquad (26)$$

and the classical vacuum part includes the Einstein–Hilbert action with cosmological constant

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R+2\Lambda)\,. \tag{27}$$

and the higher derivative terms,

$$S_{HD} = \int d^4x\sqrt{-g}\left\{a_1C^2 + a_2E + a_3\Box R + a_4R^2\right\}. \tag{28}$$

Obviously, the R^2-terms in (28) and (25) combine, that produce a well-known ambiguity in the local part of the total action with anomaly-induced contribution [15].

Compared to the previously known solutions [22–25], the expression (25) has an extra term proportional to χP_4, where χ is a new auxiliary scalar field related to the conformal anomaly. This is exactly the structure which was extensively discussed in the context of Chern–Simons extension of general relativity starting from [9] and [8]. The remarkable difference is that the field χ in (25) has higher-derivative kinetic term and also contains a mixing with the second scalar field ξ.

2.3. Brief review of other forms of the kinetic term

Since the main output of the previous consideration is the new form of the kinetic term for the Chern–Simons modified gravity, Eq. (25), it is worthwhile to give a list of the previously known kinetic terms.

The Chern–Simons gravity is usually understood as an effective theory which should be obtained from a more fundamental theory [8,9]. Consequently, the form of the kinetic term depends on the choice of the fundamental theory. In our case it is the quantum theory of matter fields (with parity violation) on classical curved background, which led us to (25). This action is different from the previously known versions, which can be presented as

$$S = \int d^4x\sqrt{-g}\Big\{-\kappa R + \frac{\alpha}{4}\,\psi\,\tilde{R}^{\mu\nu\alpha\beta}R_{\nu\mu\alpha\beta} - \frac{\beta}{2}\left[g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi + 2V(\psi)\right]\Big\} + S_{mat}\,, \tag{29}$$

where $\kappa = 1/16\pi G$, while α and β are some new constants. The Chern–Simons coupling field is ψ with a potential term $V(\psi)$, and S_{mat} is the action of matter. Also, we used the standard notation for the dual Riemann tensor

$$\tilde{R}^{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}\,R^{\mu\nu}{}_{\rho\sigma}\,. \tag{30}$$

The modified Einstein equation for the action (29) is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{\alpha}{\kappa}C_{\mu\nu} = \frac{1}{2\kappa}T_{\mu\nu}\,, \tag{31}$$

where $T_{\mu\nu}$ is the total momentum–energy tensor. The C-tensor is defined as

$$C^{\mu\nu} = \nabla_\alpha\psi\,\epsilon^{\alpha\beta\rho(\mu}\,\nabla_\rho R^{\nu)}{}_\beta + \tilde{R}^{\beta(\mu\nu)\alpha}\,\nabla_\beta\nabla_\alpha\psi\,. \tag{32}$$

The variation of (29) with respect to the scalar field yields the equation

$$\Box\psi = \frac{dV(\psi)}{d\psi} - \frac{\alpha}{4\beta}\,\tilde{R}^{\mu\nu\alpha\beta}R_{\nu\mu\alpha\beta}\,, \tag{33}$$

which is the Klein–Gordon equation with an extra Pontryagin density source.

There are two main approached, based on different choices of the constants α and β. One of them is called non-dynamical Chern–Simons gravity, when we have $\beta = 0$. Then the scalar field does not evolve dynamically, and is a field prescribed externally. This model was introduced by Jackiw and Pi in [8], where it was defined that $\psi = t/\mu$, the choice called canonical, with μ being some dimensional parameter. The boundary conditions in this theory were discussed in [13]. The development of this approach and further references can be found in the review [10].

All solutions for the non-dynamic case must satisfy the Pontryagin constraint

$$\tilde{R}^{\mu\nu\alpha\beta}R_{\nu\mu\alpha\beta} = 0\,. \tag{34}$$

This constraint limits the space of solutions of the theory. For instance, Kerr metric cannot be solution since it does not satisfy this constraint. The rotating black hole solutions have been found within approximation schemes, with certain inconsistencies discussed in the literature [26]. On the other hand, it was discussed that ghosts cannot be avoided in the non-dynamic theory [27], forcing to pay more attention to the dynamic scalar case.

The dynamic case corresponds to an arbitrary α and β in the action (29). Such a model was introduced by Smith et al. in [28], motivated by the low energy limit of string theory. The potential term was supposed to follow from the fundamental string theory, however, as usual, there is some freedom in this part. For a zero potential, $V(\psi) = 0$, there is a uniqueness theorem which ensures that in case spherically symmetric, static and asymptotically flat spacetime, the solution is given by the Schwarzschild metric [29]. In [30], the uniqueness was established for the case the Reissner–Nordstrom metrics.

Until now, there are no exact solutions for a rotating black hole and some approximate schemes are used instead. For example, in [26,31,32] the slowly rotating black hole was studied in the small-coupling limit, while [33] carried out the study for an arbitrarily large coupling.

Recently, the post-GR corrections for the dynamic model has been considered through the study of rapidly rotating black holes in the decoupling limit [34,35]. Such work is important because of the possibility to constraint the theory in the strong field regime.

The gravitational perturbations are fundamental for better understanding of the gravitational waves and the stability of solutions. In [27,36,37] the gravitational perturbations were explored for the black hole background and in [38] for the cosmological case. The issue of ghosts was studied for both non-dynamic and dynamic models. Although in both models the ghosts are present, one can avoid them for a constant scalar background in the dynamic case [27], while in the non-dynamical theory the scalar behavior is fixed. Let us mention that the works listed above were done for zero potential of the scalar field, while in the recent work [39] the gravitational perturbations with the mass term $V(\psi) = m^2\psi^2$ were considered.

Indeed, the choice of the kinetic term is non-trivial, in particular it was recognized that the Lagrangian of the kinetic term does not necessarily be of the Klein–Gordon type. Other kinetic terms were considered, e.g. the one discussed in [10] has some relation to string theory

$$S_\psi = -\frac{1}{2}\int d^4x\sqrt{-g}\Big\{\beta_1\,g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi + \beta_2\,(g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi)^2\Big\}, \tag{35}$$

where β_1 and β_2 are some constants.

An approach which is the closest one to our result (25) was developed in [40,41] and eventually used to describe the linear stability in [42]. The corresponding theory is known as Quadratic Modified Gravity, the action can be cast into the form

$$S = \int d^4x\sqrt{-g}\Big\{\kappa R + f_1(\psi)R^2 + f_2(\psi)R^2_{\mu\nu} + f_3(\psi)R^2_{\mu\nu\alpha\beta} + f_4(\psi)\tilde{R}^{\mu\nu\alpha\beta}R_{\nu\mu\alpha\beta} - \frac{\beta}{2}\Big[g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi + 2V(\psi)\Big]\Big\} + S_{mat}\,, \tag{36}$$

where $f_i(\psi)$ are some functions of the scalar field. One can easily see that in the case of (25) all these functions are linear, the coefficients are defined by the number of quantum particles and the kinetic term is more complicated and involves higher derivative and the second scalar. Also, the potential term is absent for the anomaly-induced version of the Chern–Simons gravity (25).

The situation with ghosts and tachyons in the theory (36) was discussed in [27]. It is important to note that the fields χ and ξ are auxiliary scalars, which just exist to parametrize the non-localities in the original action (18). This feature removes the need for discussing the ghosts related to the fields χ and ξ. Another way to understand this is to remember that the terms in the action (18) are at least of the third order in curvature (except R^2, which does not produce ghosts [45]). Therefore, the quantum part of induced actions presented above has no issue with ghosts, at least on the flat background.[2] Of course, the classical action behind the anomaly contains a usual C^2-term, which is known to produce ghosts. However, there are some indications that the ghost is not becoming a real particle at the energies below Planck scale [46].

3. Interpretation and applications of the parity-violating terms

The presence of imaginary parity-violating terms in the conformal anomaly (1) can be interpreted such that the existence of massless left-handed neutrino should be theoretically disfavoured [4]. This possibility looks very interesting, especially in view of experimental confirmation of neutrino oscillations. However, one has to remember that the conformal anomaly is not a directly observable physical quantity. The remarkable exception is the cosmological FRW-like solution, when the metric depends only on the conformal factor according to Eq. (9), with $\sigma = \sigma(\eta)$ and η conformal time. The trace anomaly directly affects the dynamics of $\sigma = \sigma(\eta)$. But, as far as Weyl tensor is zero for the FRW-like metric, Eq. (46) shows that the Pontryagin term is also zero for this metric. Consequently, the background cosmological solution is not affected by the presence of the new term with P_4.

In all other cases the solution (25) is not exact. This means that the effect of the P_4-dependent term can be, in principle, compensated by the conformal functional $S_c(g_{\mu\nu})$. This means that all the conclusions concerning the possible physical effects related to P_4 assume that the functional $S_c(g_{\mu\nu})$ is irrelevant. On the other hand, the general arguments presented above show that this assumption is quite reasonable. Then, we can expect that the P_4-term can be relevant for the gravitational waves on the cosmological (or other) background, and also for the physically relevant solutions such as Schwarzschild, Reissner–Nordstrom or Kerr.

The two aspects of the P_4-term in the anomaly (1) and action (25) can be relevant. The first one is that this term is parity-violating. This means that it is expected to produce a parity-odd solutions, including for the metric perturbations. As a result, one can expect the parity-odd component to emerge in the CMB spectrum. The second aspect is related to the imaginary coefficient. Let us present some considerations of these two aspects, but start from the general discussion of the solution in the presence of the Pontryagin term.

[2] This does not exclude the emergence of ghosts on other background, as it was discussed in [43] and recently in [44] in relation to the final stage of the de Sitter phase of the evolution of the ΛCDM universe.

3.1. Possible gravitational solutions

The solutions in the Chern–Simons modified theory of gravity have been extensively discussed in the literature, e.g., in the papers [40,42]. The main difference between the models which were previously considered and (25) is the form of the kinetic term for the scalar field and the presence of higher derivative terms. Let us consider in some details the simplest case of the spherically-symmetric solution, which is quite illustrative. Our purpose is not to find a new solution, but only show that the parity-violating and other higher-derivative terms in the action (25) may modify the usual Schwarzschild solution. This does not happen with the classical higher derivative terms of (28), because for the Ricci-flat background the Weyl-square term in $d = 4$ can be easily reduced to the Gauss–Bonnet topological invariant [47]. So, we can completely concentrate on the anomaly-induced part. The equations for the auxiliary fields have the form

$$\Delta_4\xi = k_1\big(E - \frac{2}{3}\Box R\big) - k_2\left(\omega C^2 + \epsilon P_4\right),$$
$$\Delta_4\chi = -k_1\big(E - \frac{2}{3}\Box R\big). \tag{37}$$

Furthermore, in the Ricci-flat case the Paneitz operator becomes simply $\Box^2$, and also one has $E = C^2 = R^2_{\mu\nu\alpha\beta}$. For the sake of simplicity, consider the possible solutions of the form [8]

$$\xi = d_1 t + f_1(r)\,, \qquad \chi = d_2 t + f_2(r)\,, \tag{38}$$

where $d_{1,2}$ are constants and $f_{1,2}$ some functions of the radial variable r. Assuming that the metric satisfies Schwarzschild solution, one has $P_4 = 0$, $R^2_{\mu\nu\alpha\beta} = 48(GM)^2/r^6$ and

$$\Box^2 f_{1,2}(r) = \frac{\alpha_{1,2}\,(GM)^2}{r^6}\,, \tag{39}$$

where $\alpha_1 = 12(k_1 - \omega k_2)$ and $\alpha_2 = -12k_1$. The general solution for the functions $f_{1,2}(r)$ corresponds to the equations (here $f = f_{1,2}$ and $\alpha = \alpha_{1,2}$) was obtained in [18],

$$\frac{df}{dr} = \frac{Br}{3} + \frac{2MB}{3} - \frac{A}{6} - \frac{\alpha}{72M} + \left(\frac{4}{3}BM^2 + \frac{C}{2M} - AM - \frac{\alpha}{24}\right)\frac{1}{r-2M} - \frac{C}{2M}\frac{1}{r} - \frac{\alpha M}{18}\frac{\ln r}{r(r-2M)} - \left(\frac{A}{2M} - \frac{\alpha}{48M^2}\right)\frac{r^2\ln r}{3(r-2M)} + \left(\frac{A}{2M} - \frac{\alpha}{48M^2}\right)\frac{(r^3 - 8M^3)\ln(r-2M)}{3r(r-2M)}\,. \tag{40}$$

Here (d, A, B, C) are constants that specify the homogeneous solution of $\Box^2 f = 0$. However, it is not necessary that the equation for the metric can be satisfied for any choice of the coefficients $(d, A, B, C)_{1,2}$. In the part which is important for our consideration, however, one can see that there is no real influence of the P_4 term, at this level. Moreover, if we assume, as an approximation, that at low energies higher derivative parity-even terms are irrelevant and only parity-odd P_4-dependent term is relevant, then the Schwarzschild solution is valid in this truncated version of the theory. Finally, Eq. (38) which is typical [18] in the higher derivative model such as (25), shows that the difference between dynamical and non-dynamical versions of the Chern–Simons modified gravity can be resolved, at least for some particular solutions.

3.2. Parity violation

What could be the effect of parity-violating term in the gravitational action? In order to answer this question, one has to remember that the most likely manifestation of the Pontryagin term is the parity violation in the gravitational waves spectrum [9]. Due to the Planck suppression the effect is going to be very weak and perhaps cannot be observed directly. However, the parity violation can eventually go to the CMB through the well-known mechanisms (see, e.g., [48,49]) and may eventually lead to the anisotropy in the metric perturbations.

3.3. Imaginary coefficient

The imaginary component of effective action is a typical phenomena in quantum field theory [50]. Usually it is related to the logarithmic structure in the form factors at the UV (the same as conformal anomaly) and indicates to the possible particle production by external field [51,52]. In order to have such a production, the energy of created particles should be smaller than the intensity of an external field. In the case of strictly massless neutrino this condition can be easily satisfied. However, some other details must be taken into account. The production of massless left-handed neutrino will be related to the fourth-derivative term $k_2 \chi \epsilon P_4$ in the effective action (25). This term is strongly suppressed by the Planck mass in the Einstein–Hilbert term, even in the inflationary period, except in the initial stable phase of the modified Starobinsky inflation model [53]. And in this special case any kind of particle production is compensated by the powerful inflation, such that the density of created particles remains negligible.

After inflation the energy of the created neutrino particles would be very small, at most of the order of the energy of the gravitational waves, since the effect is zero for the FRW-like background. During the long period of existence of the Universe there can be certain production of such particles, but there is another aspect of the problem. The neutrinos are fermions and, with a very small energy, fermionic particles should form a Fermi surface. Then the production of neutrino should be suppressed by the Pauli principle. From the quantum theory viewpoint this means the existence of the effect of quantum interaction between neutrino, which would forbid their creation. All in all, the imaginary component of the effective action cannot be probably seen as a basis of the no-go theorem forbidding theoretically massless neutrino.

4. Conclusions

We presented a simple derivation of the anomaly-induced effective action of gravity with the new parity-violating term in conformal anomaly, which was recently discovered in [4]. The integration proceeds with a minimal changes compared to the known procedure, since the new term is both topological and conformal. The result of the integration represents a new version of the well-known Chern–Simons modification of general relativity, which was extensively discussed in the literature, starting from [8] and [9]. The possible role of such a term was explored in details [10], and we cannot add much to this discussion, except to suggest a new form of kinetic term for the auxiliary scalar field, derived in (25).

Concerning the physical significance of the Pontryagin term, there is no doubt that the presence of parity-violating term in gravity is potentially very interesting [10]. The reason is that even a very small violation of the symmetry can give an observable effect. At the same time, some simple qualitative arguments show that the effect of imaginary term in the effective action and the related production of neutrino by the gravitational background should be too weak to provide a theoretical “prohibition” of the massless neutrino, as it was suggested in [4].

One more observation concerns the electromagnetic sector of the anomaly-induced action. There is no parity violation in this part of the action. However, if one could find some field (in the baryonic or dark sectors of the spectrum), which produce the parity-violating term in the conformal anomaly, the mechanism which we described in this paper would immediately generate axion with a very specific form of the kinetic term, equal to the one presented in (25).

Finally, let us say a few words about the perspectives of the new form of the Chern–Simons modified gravity (25). It is obvious that it would be interesting to check both theoretical and phenomenological consequences of this theory, starting from the solution for the rotating black hole and cosmological applications. From the QFT side, it would be interesting to see whether some (probably reduced) version of this term can be derived in other theories, in particular whether it can be met in the theory on massive neutrino at very low energies, due to the difference of the masses of the right and left components from one side and the effect of gravitational decoupling [54] from another side. Regardless of the serious technical difficulties of this program, it does not look completely impossible to be completed.

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Appendix A

The Pontryagin density is given by

$$P_4 = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta\,.\,.}{}^{\tau\lambda} R_{\mu\nu\tau\lambda}\,. \tag{41}$$

Let us prove that the Riemann tensor here can be replaced by the Weyl tensor. In 4-dimensional space we have the following relation between Riemann and Weyl tensors,

$$\begin{aligned} R_{\mu\nu\alpha\beta} = {} & C_{\mu\nu\alpha\beta} \\ & + \frac{1}{2}(R_{\mu\alpha}g_{\nu\beta} - R_{\mu\beta}g_{\nu\alpha} + R_{\nu\beta}g_{\mu\alpha} - R_{\nu\alpha}g_{\mu\beta}) \\ & - \frac{1}{6}R(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})\,. \end{aligned} \tag{42}$$

Replacing (42) into (41) we get

$$\begin{aligned} P_4 = {} & \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} C_{\alpha\beta\,.\,.}{}^{\tau\lambda} C_{\mu\nu\tau\lambda} + 2\epsilon^{\mu\nu\alpha\beta} C_{\mu\nu\lambda\beta} R^{\lambda}_{\alpha} \\ & - \frac{1}{3}\epsilon^{\mu\nu\alpha\beta} C_{\mu\nu\alpha\beta} R\,. \end{aligned} \tag{43}$$

On the other hand, Bianchi identity for the Weyl tensor

$$C_{\mu\nu\alpha\beta} + C_{\mu\beta\nu\alpha} + C_{\mu\alpha\beta\nu} = 0\,, \tag{44}$$

provide the relations

$$\epsilon^{\mu\nu\alpha\beta} C_{\mu\nu\lambda\beta} = 0\ ,\ \ \epsilon^{\mu\nu\alpha\beta} C_{\mu\nu\alpha\beta} = 0\,. \tag{45}$$

Therefore, we arrive at the desired formula,

$$P_4 = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} C_{\alpha\beta\,..}{}^{\tau\lambda} C_{\mu\nu\tau\lambda}\,, \tag{46}$$

that shows $\sqrt{-g}P_4$ to be conformal invariant.

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