



Singular inflation from generalized equation of state fluids



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ABSTRACT

We study models with a generalized inhomogeneous equation of state fluids, in the context of singular inflation, focusing to so-called Type IV singular evolution. In the simplest case, this cosmological fluid is described by an equation of state with constant w , and therefore a direct modification of this constant w fluid is achieved by using a generalized form of an equation of state. We investigate from which models with generalized phenomenological equation of state, a Type IV singular inflation can be generated and what the phenomenological implications of this singularity would be. We support our results with illustrative examples and we also study the impact of the Type IV singularities on the slow-roll parameters and on the observational inflationary indices, showing the consistency with Planck mission results. The unification of singular inflation with singular dark energy era for specific generalized fluids is also proposed.

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1. Introduction

One of the most astonishing surprises that the scientific community experienced in the late 90's was the observationally verified late-time acceleration of the Universe [1], an observation which was contrary to any up to that date perception or expectation for the Universe's late-time evolution. Since then, considerable amount of research was conducted towards the understanding and explanation of this late-time acceleration. In addition, a desirable feature that a complete theory of cosmological evolution should have is the description of the early-time acceleration, known as inflationary era [2,3], and of the late-time acceleration using the same theoretical framework. One of the most successful approaches that describe late-time and early-time acceleration in unified manner [4] is provided by $F(R)$ theories [5], which serve as modifications of the standard Einstein–Hilbert gravity. It is very interesting that modification of gravity may be considered as addition of (geometric) terms to the phenomenological equation of state of corresponding generalized fluid. In fact, it is quite well-known that generalized (imperfect) fluids EoS [7] which contain inhomogeneous terms may successfully describe the Universe acceleration. The homogeneous modifications assume the inclusion of terms in the EoS which depend on the effective energy density, and the inhomogeneous modifications assume terms that depend explicitly on the Hubble rate or its higher derivatives, and as was demonstrated in [7], in the context of inhomogeneous phenomenological EoS theories, it is possible to describe even phantom evolution without introducing a scalar field with negative kinetic energy. Apart from this appealing feature of phenomenological EoS theories, a strong motivation to use and study these comes from the fact that inhomogeneous terms may be understood as the time-dependent bulk or shear viscosity [8] and also symmetry considerations indicate such modifications of EoS [9]. Such generalized fluids may be used for the construction of inflationary era (see, for instance [10]). Finally and most importantly, it can be proven that in the context of modified gravity theories, the resulting EoS of the geometric fluid is modified in the fashion we described above [5,6].

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It is common in the context of these theories, however, that various types of finite time singularities frequently occur. It is remarkable that such finite-time singularities are universal, as they may occur after early-time acceleration as well as after late-time acceleration. The purpose of this article is to study the Universe's cosmological evolution using a phenomenological EoS for the dark fluid, that leads to, or is responsible for, a Type IV finite time singularities. The interest to this specific singularity is owing to the fact that the Universe's evolution continues smoothly after passing through this singularity! This argument is strongly supported by the fact that no geodesics incompleteness occurs for sudden and this type of singularities, so in principle the evolution is not abruptly interrupted by the singular behavior of some physical quantities, such as the higher derivatives of the Hubble rate. The study of realistic cosmological singularities was initiated sometime ago [11]. The very interesting finite time future singularities are known as sudden singularities or Type II ones (for later studied on these, see [12–14]). Also the finite time cosmological singularities were firstly classified in [15], with the classification taking into account whether physical quantities like the scale factor, the effective energy density, or pressure, are finite or infinite at certain spacetime points. The Type IV singularity is the most “mild” singularity among the four types of singularities, with the mild referring to the fact that these are not crushing type singularities. For recent studies on these singularities see [16–18], where details can be found for the various types of cosmological evolution that these can generate or modify.

The purpose of this letter is to demonstrate at first time that singular inflation induced by generalized fluids is quite possible. The unification of singular inflation with singular dark energy era in the Universe filled with generalized fluid is also proposed. The paper outline is: In Section 2, we present some essential information for the theoretical framework we shall use and we study in detail how a singular evolution can be produced by generalized EoS models and we also provide some concrete examples. In Section 3, we address the slow-roll behavior issue of these generalized EoS theories and in Section 4 the conclusions follow.

2. Theoretical framework and analysis of singular generalized EoS models

To start with, consider a perfect fluid coupled with the standard Einstein–Hilbert gravity, with the corresponding Friedmann–Robertson–Walker (FRW) equations being equal to,

$$\rho = \frac{3}{\kappa^2} H^2, \quad p = -\frac{1}{\kappa^2} (3H^2 + 2\dot{H}), \quad (1)$$

where it is assumed that the background metric is a flat FRW metric of the following form,

$$ds^2 = -dt^2 + a^2(t) \sum_i dx_i^2. \quad (2)$$

In Eq. (1), H denotes the Hubble rate, as usual, $H(t) = \dot{a}(t)/a(t)$. For a given cosmological evolution in terms of the Hubble rate $H = H(t)$, the right hand side of the two equations appearing in Eq. (1) are solely functions of t ,

$$\rho = f_\rho(t), \quad p = f_p(t). \quad (3)$$

Then, by solving the first equation with respect to the cosmic time t , we get, $t = f_\rho^{-1}(\rho)$, and by substituting the resulting expression into the second equation of Eq. (3), we obtain the functional form of the EoS, namely,

$$p = f_p(f_\rho^{-1}(\rho)). \quad (4)$$

In this way we can obtain the exact form of the effective EoS, which we denote as w_{eff} . It is worth recalling in brief the essential features of the homogeneous and inhomogeneous phenomenological EoS theories, the complete presentation of which can be found in [7]. In the context of inhomogeneous phenomenological EoS theories the EoS is modified as follows,

$$p = -\rho - f(\rho) + G(H), \quad (5)$$

and when the function $G(H)$ is zero, this corresponds to the homogeneous EoS theory. A much more general functional dependence of the effective pressure as a function of the effective energy density and the Hubble rate, is given by the following functional form,

$$p = f(\rho, H). \quad (6)$$

In Eqs. (5) and (6), the functions f and G are arbitrary functions of their arguments.

The focus in this letter is to study the phenomenological EoS theories of Eq. (6), for a Type IV cosmological evolution as it occurs in the inflationary epoch. It is worth recalling in brief the classification of finite time cosmological singularities, as in Ref. [15]:

- Type I (“Big Rip”): When $t \rightarrow t_s$, the scale factor a , the effective energy density ρ_{eff} , and the effective pressure p_{eff} diverge, namely $a \rightarrow \infty$, $\rho_{\text{eff}} \rightarrow \infty$, and $|p_{\text{eff}}| \rightarrow \infty$. For details on the Big Rip finite singularity, the reader is referred to [19,15,20].
- Type II (“sudden”): When $t \rightarrow t_s$, although both of the scale factor and the effective energy density are finite, that is, $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \rho_s$, the effective pressure diverges, namely $|p_{\text{eff}}| \rightarrow \infty$. See [12,13] for further analysis on this type of singularities.
- Type III: When $t \rightarrow t_s$, although the scale factor is finite, $a \rightarrow a_s$, both of the effective energy density and the effective pressure diverge, $\rho_{\text{eff}} \rightarrow \infty$, $|p_{\text{eff}}| \rightarrow \infty$.
- Type IV: When $t \rightarrow t_s$, all of the scale factor, the effective energy density, and the effective pressure are finite, that is, $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \rho_s$, $|p_{\text{eff}}| \rightarrow p_s$, but the higher derivatives of the Hubble rate diverge. For a detailed account on this singularity, see [15].

Here the effective energy density ρ_{eff} , and the effective pressure p_{eff} are defined by $\rho_{\text{eff}} \equiv (3/\kappa^2) H^2$, and $p_{\text{eff}} \equiv -(1/\kappa^2) (2\dot{H} + 3H^2)$. A quite general form of the Hubble rate that can lead to Type II and Type IV singular cosmological evolution, is the following,

$$H(t) = f_1(t) + f_2(t) (t_s - t)^\alpha, \quad (7)$$

where the functions $f_1(t)$ and $f_2(t)$ are considered to be smooth and differentiable functions of t . The Type II singularity occurs when the constant parameter α is restricted to take the values, $0 < \alpha < 1$, while when $\alpha > 1$, the cosmological evolution develops a Type IV singularity. In the following, and in order to avoid inconsistencies, we assume that the parameter α appearing in Eq. (7), takes the following form,

$$\alpha = \frac{n}{2m + 1}, \quad (8)$$

where n and m are arbitrarily chosen positive integers. In addition, a modified version of the Hubble rate appearing in Eq. (7), is given below,

$$H(t) = f_1(t) + f_2(t) |t_s - t|^\alpha, \quad (9)$$

in which case, α can take more general values, without assuming the restricted form given in Eq. (8), but we study here only the case for which α has the form given in Eq. (8). In the following we shall investigate which homogeneous and inhomogeneous EoS can generate the cosmological evolution appearing in Eq. (7), with special emphasis given in the Type IV singularity. In general, it is difficult to find an explicit form of the EoS, for complicated forms of the functions $f_1(t)$ and $f_2(t)$, so let us start with a simple example describing a Type IV evolution, with the functions $f_1(t)$ and $f_2(t)$ being chosen as, $f_1(t) = 0$ and $f_2(t) = f_0$, where f_0 is an arbitrary positive constant. By using these values and by substituting in Eq. (1) the Hubble rate of Eq. (7), the effective energy density ρ and the effective pressure p read,

$$\rho = \frac{3f_0^2}{\kappa^2} (t_s - t)^{2\alpha}, \quad p = -\frac{1}{\kappa^2} \left(3f_0^2 (t_s - t)^{2\alpha} + 2\alpha f_0 (t_s - t)^{\alpha-1} \right), \quad (10)$$

By using the first equation of Eq. (10), we can solve it in terms of $t_s - t$ and therefore by substituting in the effective pressure expression, we obtain in explicit form of the EoS, which reads (see also [15]),

$$p = -\rho - 2 \cdot 3^{-\frac{\alpha-1}{2\alpha}} \alpha \kappa^{-\frac{\alpha+1}{\alpha}} f_0^{2-\alpha} \rho^{\frac{\alpha-1}{2\alpha}}. \quad (11)$$

The exact type of the finite time singularities that may appear in this cosmological evolution may be found by using the explicit forms of the effective energy density ρ and of the effective pressure p , as these appear in Eq. (10). By introducing the parameter $\tilde{\alpha} \equiv \frac{\alpha-1}{2\alpha}$, the cosmological evolution of the phenomenological EoS (11), as described by the energy density and pressure appearing in Eq. (10), has the following singularity structure:

- In the case that $\tilde{\alpha} > 1$, or in terms of α , $-1 < \alpha < -\frac{1}{2}$, a Type III singularity occurs.
- In the case that $\frac{1}{2} < \tilde{\alpha} < 1$, that is, $\alpha < -1$, a Type I singularity appears.
- In the case that $0 < \tilde{\alpha} < \frac{1}{2}$, or equivalently, $\alpha > 1$, a Type IV singularity occurs.
- In the case that $\tilde{\alpha} < 0$, or in terms of α , $-\frac{1}{2} < \alpha < 1$, a Type II singularity occurs.

The form of the EoS appearing in Eq. (11), can be viewed as a homogeneous phenomenological EoS theory of the form (5), with $f(\rho)$ being equal to,

$$f(\rho) = -2 \cdot 3^{-\frac{\alpha-1}{2\alpha}} \alpha \kappa^{-\frac{\alpha+1}{\alpha}} f_0^{2-\alpha} \rho^{\frac{\alpha-1}{2\alpha}} \quad (12)$$

and $G(H) = 0$. In addition, the EoS appearing in Eq. (11) can also be viewed as an inhomogeneous phenomenological EoS theory of the form (5), by using the fact that ρ can be written in terms of the Hubble rate H^2 in the way dictated by Eq. (1). By doing so, the EoS can be written in the following form,

$$p = -\rho - \frac{2\alpha}{\kappa^2} f_0^{2-\alpha} H^{\frac{\alpha-1}{\alpha}}, \quad (13)$$

which is of the form given in Eq. (5), with $f(\rho) = 0$ and with $G(H)$ being equal to,

$$G(H) = -\frac{2\alpha}{\kappa^2} f_0^{2-\alpha} H^{\frac{\alpha-1}{\alpha}}. \quad (14)$$

We may also consider the following model,

$$H(t) = h_0 \left\{ \left(\frac{t-t_0}{t_1} \right)^{-2n} + 1 \right\}^{-\frac{\alpha}{2n}}, \quad (15)$$

where h_0 , t_0 , t_1 , α are constants and we assume $h_0 > 0$, $n > 0$ and also that $0 < \alpha < 1$. $\alpha > 1$. When $t \rightarrow \pm\infty$, we find $H(t)$ becomes a constant $H(t) \rightarrow h_0$ and when $t \sim t_0$, a Type IV singularity occurs, since the Hubble rate behaves as, $H(t) \sim h_0 \left(\frac{t-t_0}{t_1} \right)^\alpha$. The corresponding effective energy density and effective pressure for the Hubble rate (15) are equal to,

$$\begin{aligned} \rho &= \frac{3h_0^2}{\kappa^2} \left\{ \left(\frac{t-t_0}{t_1} \right)^{-2n} + 1 \right\}^{-\frac{\alpha}{n}}, \\ p &= -\frac{3h_0^2}{\kappa^2} \left\{ \left(\frac{t-t_0}{t_1} \right)^{-2n} + 1 \right\}^{-\frac{\alpha}{n}} - \frac{2\alpha h_0}{\kappa^2 t_1} \left\{ \left(\frac{t-t_0}{t_1} \right)^{-2n} + 1 \right\}^{-\frac{\alpha}{2n}-1} \left(\frac{t-t_0}{t_1} \right)^{-2n-1}. \end{aligned} \tag{16}$$

Consequently, by making use of Eq. (4), we obtain the following EoS,

$$p = -\rho - \frac{2\alpha h_0}{\kappa^2 t_1} \left(\frac{\kappa^2 \rho}{3h_0^2} \right)^{\frac{1}{2} + \frac{n}{\alpha}} \left\{ \left(\frac{\kappa^2 \rho}{3h_0^2} \right)^{-\frac{n}{\alpha}} - 1 \right\}^{1 + \frac{1}{2n}}. \tag{17}$$

Now consider the case in which the Hubble rate is given by,

$$H(t) = f_0(t-t_1)^\alpha + c_0(t-t_2)^\beta, \tag{18}$$

with c_0 and f_0 constant and positive parameters, and $\alpha, \beta > 1$, so that the cosmological evolution has two Type IV singularities at $t = t_1$ and $t = t_2$. We may choose t_1 to be at the end of the inflationary era and t_2 to be at late-time. In principle, it is quite difficult to obtain the exact form of the EoS for the Hubble rate (18), since it is quite difficult to solve the equation $\rho \sim H^2(t)$ with respect to t . However we can find an approximate form of the EoS, near the two Type IV singularities. Before going into the details of this approximation, we quote here the cosmic time dependence of the effective energy and of the effective pressure for the Hubble rate (18), which are,

$$\begin{aligned} \rho &= \frac{3f_0^2(t-t_1)^{2\alpha}}{\kappa^2} + \frac{6c_0f_0(t-t_1)^\alpha(t-t_2)^\beta}{\kappa^2} + \frac{3c_0^2(t-t_2)^{2\beta}}{\kappa^2}, \\ p &= -\frac{3f_0^2(t-t_1)^{2\alpha}}{\kappa^2} - \frac{6c_0f_0(t-t_1)^\alpha(t-t_2)^\beta}{\kappa^2} - \frac{3c_0^2(t-t_2)^{2\beta}}{\kappa^2} - \frac{2f_0(t-t_1)^{-1+\alpha}\alpha}{\kappa^2} - \frac{2c_0(t-t_2)^{-1+\beta}\beta}{\kappa^2}. \end{aligned} \tag{19}$$

So at the vicinity of the early-time Type IV singularity, these read,

$$\rho = \frac{3c_0^2(t-t_2)^{2\beta}}{\kappa^2}, \quad p = -\frac{3c_0^2(t-t_2)^{2\beta}}{\kappa^2} - \frac{2c_0(t-t_2)^{-1+\beta}\beta}{\kappa^2}, \tag{20}$$

so the EoS takes the approximate form,

$$p = -\rho - \frac{2c_0\beta}{\kappa^2} \left(\frac{\rho\kappa^2}{3c_0^2} \right)^{\frac{\beta-1}{2\beta}}. \tag{21}$$

The resulting physical situation is quite appealing, since it seems that the late-time singularity controls the early-time EoS for the inhomogeneous phenomenological EoS theory that generates the Hubble rate (18), near the of course early-time singularity. In fact, the resulting inhomogeneous phenomenological EoS near the early-time singularity is controlled solely from the late-time singularity, since the terms $\sim (t-t_1)$ vanish for a Type IV early-time singularity. Of course, for other types of singularities, this may not occur, so only the Type IV case has this interesting feature. The same applies for the late-time singularity, so near the late-time Type IV singularity the effective energy density and effective pressure are,

$$\rho = \frac{3c_0^2(t-t_1)^{2\alpha}}{\kappa^2}, \quad p = -\frac{3c_0^2(t-t_1)^{2\alpha}}{\kappa^2} - \frac{2c_0(t-t_1)^{-1+\alpha}\alpha}{\kappa^2}, \tag{22}$$

so that the corresponding EoS reads,

$$p = -\rho - \frac{2c_0\alpha}{\kappa^2} \left(\frac{\rho\kappa^2}{3c_0^2} \right)^{\frac{\alpha-1}{2\alpha}}. \tag{23}$$

Therefore, the resulting picture is that the EoS near the late-time singularity is solely controlled from the early-time Type IV singularity, which is a quite interesting feature. So by suitably choosing the parameters, this effect can be quite large or even negligible. For a relevant study in which this phenomenology also occurs, see also [18].

Another interesting model with the property that it provides a unified description of inflation at early-time with a late-time acceleration evolution with a Type IV singularity occurring at late-time. The Hubble rate of this model is given below,

$$H(t) = \frac{f_1}{\sqrt{t^2 + t_0^2}} + \frac{f_2 t^2 (-t + t_1)^\alpha}{t^4 + t_0^4} + f_3 (-t + t_2)^\beta. \tag{24}$$

Notice that the late-time singularity can even occur at present time. For a detailed account on the consequences of this Type IV singularity occurring at present time, see [16]. In Eq. (24), the parameters $\alpha, \beta, t_0, f_1, f_2,$ and f_3 are chosen to be positive constants so that we ensure that $H(t) > 0$. Then, the effective energy density ρ and the effective pressure p are equal to,

$$\rho = \frac{3}{\kappa^2} \left\{ \frac{f_1}{\sqrt{t^2 + t_0^2}} + \frac{f_2 t^2 (-t + t_1)^\alpha}{t^4 + t_0^4} + f_3 (-t + t_2)^\beta \right\}^2,$$

$$p = -\frac{3}{\kappa^2} \left\{ \frac{f_1}{\sqrt{t^2 + t_0^2}} + \frac{f_2 t^2 (-t + t_1)^\alpha}{t^4 + t_0^4} + f_3 (-t + t_2)^\beta \right\}^2 - \frac{2}{\kappa^2} \left\{ -\frac{f_1 t}{(t^2 + t_0^2)^{3/2}} - \frac{4 f_2 t^5 (-t + t_1)^\alpha}{(t^4 + t_0^4)^2} + \frac{2 f_2 t (-t + t_1)^\alpha}{t^4 + t_0^4} - \frac{f_2 t^2 (-t + t_1)^{-1+\alpha} \alpha}{t^4 + t_0^4} - f_3 \beta (-t + t_2)^{-1+\beta} \right\}. \tag{25}$$

The cosmic dark fluid scenario with effective pressure and energy density as in Eq. (25) can describe a plethora of cosmological evolutions. For example, t_1 may correspond to early-time and t_2 at late-time, so if $\alpha > 1$ and $\beta > 1$, a Type IV singularity occurs at both early- and late-time, as in the previous example. Let us briefly repeat the same analysis as in the previous case, in order to find an analytic approximation for the EoS near the singularities. Consider first the case that $t \simeq t_1$, so the physical system is considered to be near the early-time Type IV singularity. Then, the effective energy density and pressure become approximately equal to,

$$\rho \simeq \frac{3 f_1^2}{(t^2 + t_0^2) \kappa^2} + \frac{6 f_1 f_3 (-t + t_2)^\beta}{\sqrt{t^2 + t_0^2} \kappa^2} + \frac{3 f_3^2 (-t + t_2)^{2\beta}}{\kappa^2},$$

$$p \simeq \frac{2 f_1 t}{(t^2 + t_0^2)^{3/2} \kappa^2} - \frac{3 f_1^2}{(t^2 + t_0^2) \kappa^2} - \frac{6 f_1 f_3 (-t + t_2)^\beta}{\sqrt{t^2 + t_0^2} \kappa^2} - \frac{3 f_3^2 (-t + t_2)^{2\beta}}{\kappa^2} + \frac{2 f_3 (-t + t_2)^{-1+\beta} \beta}{\kappa^2}, \tag{26}$$

and as is obvious, the late-time Type IV singularity controls the effective energy density and effective pressure of the cosmological dark fluid, near the early-time Type IV singularity. In order to see how the late-time singularity controls the EoS of the phenomenological theory, let us further simplify the expressions appearing in Eq. (26), by taking into account that $t \ll t_2$, in which case the effective energy density and effective pressure read,

$$\rho \simeq \frac{3 f_3^2 (-t + t_2)^{2\beta}}{\kappa^2} \simeq \frac{3 f_3^2 t_2^{2\beta}}{\kappa^2}, \quad p \simeq -\frac{3 f_3^2 (-t + t_2)^{2\beta}}{\kappa^2} \simeq -\frac{3 f_3^2 t_2^{2\beta}}{\kappa^2}, \tag{27}$$

so that the EoS reads, $p \simeq -\rho$. Also notice that, owing to the appearance of the parameter t_2 , the late-time singularity drastically affects the early-time EoS and acceleration. The latter, is due to the fact that the early-time evolution is nearly de Sitter acceleration (since $p \simeq -\rho$).

Conversely, at late-time and near the future Type IV singularity, that is, when $t \simeq t_2$, the corresponding effective energy density and pressure of the phenomenological EoS theory are,

$$\rho \simeq \frac{3 f_1^2}{(t^2 + t_0^2) \kappa^2} + \frac{6 f_0 f_1 t^2 (-t + t_1)^\alpha}{\sqrt{t^2 + t_0^2} (t^4 + t_0^4) \kappa^2} + \frac{3 f_0^2 t^4 (-t + t_1)^{2\alpha}}{(t^4 + t_0^4)^2 \kappa^2},$$

$$p \simeq \frac{2 f_1 t}{(t^2 + t_0^2)^{3/2} \kappa^2} - \frac{3 f_1^2}{(t^2 + t_0^2) \kappa^2} + \frac{8 f_0 t^5 (-t + t_1)^\alpha}{(t^4 + t_0^4)^2 \kappa^2} - \frac{4 f_0 t (-t + t_1)^\alpha}{(t^4 + t_0^4) \kappa^2} - \frac{6 f_0 f_1 t^2 (-t + t_1)^\alpha}{\sqrt{t^2 + t_0^2} (t^4 + t_0^4) \kappa^2} - \frac{3 f_0^2 t^4 (-t + t_1)^{2\alpha}}{(t^4 + t_0^4)^2 \kappa^2} + \frac{2 f_0 t^2 (-t + t_1)^{-1+\alpha} \alpha}{(t^4 + t_0^4) \kappa^2}, \tag{28}$$

so practically, we could say that the early-time Type IV singularity controls the late-time behavior of the phenomenological EoS theory. Let us see however the extent of this control explicitly. Since we assumed that $t \gg t_1$, the effective energy density and pressure appearing in Eq. (28) are very much simplified, and by keeping only leading order terms, these become approximately equal to,

$$\rho \simeq \frac{3 f_0^2 t^4 (-t + t_1)^{2\alpha}}{(t^4 + t_0^4)^2 \kappa^2}, \quad p \simeq -\frac{3 f_0^2 t^4 (-t + t_1)^{2\alpha}}{(t^4 + t_0^4)^2 \kappa^2}, \tag{29}$$

and hence it easily follows that in this case, the EoS is $p \simeq -\rho$, so we have late-time de Sitter acceleration with $w_{\text{eff}} \simeq -1$. Therefore, the contribution of the early-time singularity is not so important in this case. Another interesting example with two different Type IV singularities is described by the following cosmological evolution,

$$H(t) = f_0 + c (t - t_1)^\alpha (t - t_2)^\beta, \tag{30}$$

in which case, if $\alpha, \beta > 1$, Type IV singularities occur at $t = t_1$ and at $t = t_2$. The corresponding effective energy density and pressure are equal to,

$$\begin{aligned}\rho &= \frac{3c^2}{\kappa^2} + \frac{6cf_0(-t+t_1)^\alpha(-t+t_2)^\beta}{\kappa^2} + \frac{3f_0^2(-t+t_1)^{2\alpha}(-t+t_2)^{2\beta}}{\kappa^2}, \\ p &= -\frac{3c^2}{\kappa^2} - \frac{6cf_0(-t+t_1)^\alpha(-t+t_2)^\beta}{\kappa^2} - \frac{3f_0^2(-t+t_1)^{2\alpha}(-t+t_2)^{2\beta}}{\kappa^2} \\ &\quad + \frac{2f_0(-t+t_1)^{-1+\alpha}(-t+t_2)^\beta\alpha}{\kappa^2} + \frac{2f_0(-t+t_1)^\alpha(-t+t_2)^{-1+\beta}\beta}{\kappa^2}.\end{aligned}\quad (31)$$

At both singular points, the effective energy and pressure are exactly equal to,

$$\rho = \frac{3c^2}{\kappa^2}, \quad p = -\frac{3c^2}{\kappa^2}, \quad (32)$$

which implies that the EoS is exactly equal to minus one, since $p = -\rho$. We have to note that in all the above paradigms, the results can drastically change, if the singularities we considered are not Type IV.

The converse procedure is possible to produce a Type IV singular cosmological evolution, from a given phenomenological EoS theory. Indeed, consider a homogeneous phenomenological EoS theory, with the EoS being of the form,

$$p = -\rho + f(\rho), \quad (33)$$

with $f(\rho)$ being equal to,

$$f(\rho) = A\rho^\alpha. \quad (34)$$

Then, by using the energy conservation law, we easily obtain that,

$$\rho = (t-t_0)^{\frac{2}{1-\alpha}} \left(\frac{\sqrt{3}\kappa A}{2} \right)^{\frac{2}{1-\alpha}}. \quad (35)$$

Correspondingly, the effective pressure as a function of time reads,

$$\begin{aligned}p &= -\frac{\left(\frac{9}{16}\right)^{\frac{\alpha}{1-2\alpha}}(t-t_0)^{\frac{4\alpha}{1-2\alpha}}(A\kappa)^{\frac{2}{1-2\alpha}}\left(\left(\frac{3}{4}\right)^{\frac{1}{1-2\alpha}}(t-t_0)^{\frac{2}{1-2\alpha}}(A\kappa)^{\frac{2}{1-2\alpha}}\right)^{-2\alpha}}{(1-2\alpha)^2\kappa^2} \\ &\quad \times \frac{\left(\left(\frac{3}{4}\right)^{\frac{1}{1-2\alpha}}(t-t_0)^{\frac{2}{1-2\alpha}}(A\kappa)^{\frac{2}{1-2\alpha}} + \left(\left(\frac{3}{4}\right)^{\frac{1}{1-2\alpha}}(t-t_0)^{\frac{2}{1-2\alpha}}(A\kappa)^{\frac{2}{1-2\alpha}}\right)^\alpha\right)}{(1-2\alpha)^2\kappa^2},\end{aligned}\quad (36)$$

so when $0 < \alpha < \frac{1}{2}$, a Type IV singularity occurs in the cosmological evolution, at $t = t_0$.

3. Slow-roll parameters for singular generalized EoS models

In all the above cases, we investigated the general behavior of various homogeneous or inhomogeneous phenomenological EoS theories, in the presence of a Type IV singularity during the cosmological evolution. However, as was stressed in [16], the presence of a Type IV singularity during the inflationary era may have dramatic consequences on the inflationary observational indices. It is therefore compelling to study the behavior of these observational indices in the presence of a Type IV singularity. We shall stay at a qualitative level analysis, but we shall attempt to compare some results with the recent Planck observational data [21]. This analysis however is strongly model dependent, so in this letter we will just highlight the most important qualitative implications of the Type IV singularities.

The full analysis on the derivation of the slow-roll parameters for a perfect fluid phenomenological EoS theory can be found in Refs. [6], and the key point is that the FRW equations appearing in Eq. (1) can be expressed in terms of the e -folding number N , as follows,

$$\rho = \frac{3}{\kappa^2} (H(N))^2, \quad p(N) + \rho(N) = -\frac{2H(N)H'(N)}{\kappa^2}, \quad (37)$$

where $H'(N) = dH/dN$. Assuming that the EoS has the following general form,

$$p(N) = -\rho_{\text{matter}}(N) + \tilde{f}(\rho(N)), \quad (38)$$

the second equation in (37), takes the following form,

$$\tilde{f}(\rho(N)) = -\frac{2H(N)H'(N)}{\kappa^2}. \quad (39)$$

The energy density and the pressure satisfy the conservation law,

$$\rho'(N) + 3H(N)(\rho(N) + p(N)) = 0, \quad (40)$$

where $\rho'(N) = d\tilde{f}(\rho(N))/dN$ and in view of Eq. (39), this conservation law becomes,

$$\rho'(N) + 3\tilde{f}(\rho(N)) = 0. \quad (41)$$

Combining Eqs. (41) and (38), we obtain,

$$\frac{2}{\kappa^2} \left[(H'(N))^2 + H(N) + H''(N) \right] = 3\tilde{f}'(\rho)f(\rho), \tag{42}$$

where in this case $\tilde{f}'(\rho(N))$ stands for $\tilde{f}'(\rho(N)) = d\tilde{f}(\rho)/d\rho$. Consequently, we may express the slow-roll parameters as functions only of $\rho(N)$ and $\tilde{f}(\rho(N))$ or equivalently as functions of $H(N)$ and it's derivatives (for details see [6]). For the present purposes we shall use the slow-roll parameters as functions of $H(N)$, but later on we shall use the explicit form of the function $\tilde{f}(\rho(N))$, in order to have comparison with observational data coming from Planck [21]. The slow-roll parameters ϵ , η and ξ are given in terms of $H(N)$ as follows,

$$\begin{aligned} \epsilon &= -\frac{H(N)}{4H'(N)} \left[\frac{\frac{6H'(N)}{H(N)} + \frac{H''(N)}{H(N)} + \left(\frac{H'(N)}{H(N)}\right)^2}{3 + \frac{H'(N)}{H(N)}} \right]^2, \\ \eta &= -\frac{1}{2} \left(3 + \frac{H'(N)}{H(N)}\right)^{-1} \left[\frac{9H'(N)}{H(N)} + \frac{3H''(N)}{H(N)} + \frac{1}{2} \left(\frac{H'(N)}{H(N)}\right)^2 - \frac{1}{2} \left(\frac{H''(N)}{H'(N)}\right)^2 + \frac{3H''(N)}{H'(N)} + \frac{H'''(N)}{H'(N)} \right], \\ \xi^2 &= \frac{\frac{6H'(N)}{H(N)} + \frac{H''(N)}{H(N)} + \left(\frac{H'(N)}{H(N)}\right)^2}{4 \left(3 + \frac{H'(N)}{H(N)}\right)^2} \left[\frac{3H(N)H'''(N)}{H'(N)^2} + \frac{9H'(N)}{H(N)} - \frac{2H(N)H''(N)H'''(N)}{H'(N)^3} + \frac{4H''(N)}{H(N)} \right. \\ &\quad \left. + \frac{H(N)H''(N)^3}{H'(N)^4} + \frac{5H'''(N)}{H'(N)} - \frac{3H(N)H''(N)^2}{H'(N)^3} - \left(\frac{H''(N)}{H'(N)}\right)^2 + \frac{15H''(N)}{H'(N)} + \frac{H(N)H''''(N)}{H'(N)^2} \right]. \end{aligned} \tag{43}$$

It is convenient to have the slow-roll parameters as explicit functions of the cosmic time, in order to examine the various forms of the Hubble rate we presented in this letter, that actually lead to a Type IV singularity. In this way, we will explicitly study how the singularity that occurs at a finite time t_s , affects the slow-roll parameters and therefore the approximation itself. These expressions are given below,

$$\begin{aligned} \epsilon &= -\frac{H^2}{4\dot{H}} \left(\frac{6\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \right)^2 \left(3 + \frac{\dot{H}}{H^2} \right)^{-2}, \\ \eta &= -\frac{1}{2} \left(3 + \frac{\dot{H}}{H^2} \right)^{-1} \left(\frac{6\dot{H}}{H^2} + \frac{\dot{H}^2}{2H^4} - \frac{\ddot{H}}{H^3} - \frac{\dot{H}^4}{2H^4} + \frac{\dot{H}^2\ddot{H}}{H^5} - \frac{\ddot{H}^2}{2H^2} + \frac{3\ddot{H}}{H\dot{H}} + \frac{\ddot{H}}{H^2\dot{H}} \right), \\ \xi^2 &= \frac{1}{4} \left(\frac{6\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \right) \left(3 + \frac{\dot{H}}{H^2} \right)^{-1} \left(\frac{9\ddot{H}}{H\dot{H}} + \frac{3\ddot{H}}{H^2} + \frac{2\ddot{H}}{H^2\dot{H}} + \frac{4\dot{H}^2}{H^2\dot{H}^2} - \frac{\ddot{H}\ddot{H}}{H\dot{H}^3} - \frac{3\ddot{H}^2}{\dot{H}^3} + \frac{\ddot{H}^3}{H\dot{H}^4} + \frac{\ddot{H}}{H\dot{H}^2} \right). \end{aligned} \tag{44}$$

Assume for the moment that the Hubble rate $H(t)$ is given by the very general form of Eq. (7), and therefore when $\alpha > 1$, a Type IV singularity is realized. Also assume that the function $f_1(t)$ is constrained in such a way so that $f_1(t_s)$, $f_1'(t_s)$, and $f_1''(t_s)$ do not vanish. Then, the slow-roll parameters at the vicinity of the Type IV singularity $t \sim t_s$, behave as follows,

$$\begin{aligned} \epsilon &\sim \begin{cases} -\frac{f_1(t_s)^2}{4f_1'(t_s)} \left(\frac{6\dot{f}_1(t_s)f_1(t_s)}{f_1(t_s)^2} + \frac{\ddot{f}_1(t_s)}{f_1(t_s)^3} \right)^2 \left(3 + \frac{\dot{f}_1(t_s)}{f_1(t_s)^2} \right)^{-2}, & \text{when } \alpha > 2 \\ -\frac{f_1(t_s)^2}{4f_1'(t_s)} f_2(t_s)\alpha(\alpha-1)(t_s-t)^{\alpha-2} \left(3 + \frac{\dot{f}_1(t_s)}{f_1(t_s)^2} \right)^{-2}, & \text{when } 2 > \alpha > 1 \end{cases}, \\ \eta &\sim \begin{cases} -\frac{1}{2} \left(3 + \frac{\dot{f}_1(t_s)}{f_1(t_s)^2} \right)^{-1} \\ \times \left(\frac{6\dot{f}_1(t_s)}{f_1(t_s)} + \frac{\dot{f}_1(t_s)^2}{2f_1(t_s)^4} - \frac{\ddot{f}_1(t_s)}{f_1(t_s)^3} - \frac{\dot{f}_1(t_s)^4}{2f_1(t_s)^4} + \frac{\dot{f}_1(t_s)^2\ddot{f}_1(t_s)}{f_1(t_s)^5} - \frac{\ddot{f}_1(t_s)^2}{2f_1(t_s)^2} + \frac{3\ddot{f}_1(t_s)}{f_1(t_s)\dot{f}_1(t_s)} + \frac{\ddot{f}_1(t_s)}{f_1(t_s)^2\dot{f}_1(t_s)} \right), & \text{when } \alpha > 3 \\ -\frac{1}{2} \left(3 + \frac{\dot{f}_1(t_s)}{f_1(t_s)^2} \right)^{-1} \frac{f_2\alpha(\alpha-1)(\alpha-2)}{f_1(t_s)^2\dot{f}_1(t_s)} (t_s-t)^{\alpha-3}, & \text{when } 3 > \alpha > 1 \end{cases}, \\ \xi^2 &\sim \begin{cases} \frac{1}{4} \left(\frac{6\dot{f}_1(t_s)}{f_1(t_s)^2} + \frac{\ddot{f}_1(t_s)}{f_1(t_s)^3} \right) \left(3 + \frac{\dot{f}_1(t_s)}{f_1(t_s)^2} \right)^{-1} \\ \times \left(\frac{9\ddot{f}_1(t_s)}{f_1(t_s)\dot{f}_1(t_s)} + \frac{3\ddot{f}_1(t_s)}{f_1(t_s)^2} + \frac{2\ddot{f}_1(t_s)}{f_1(t_s)^2\dot{f}_1(t_s)} + \frac{4\dot{f}_1(t_s)^2}{f_1(t_s)^2\dot{f}_1(t_s)^2} - \frac{\ddot{f}_1(t_s)\ddot{f}_1(t_s)}{f_1(t_s)\dot{f}_1(t_s)^3} \right. \\ \left. - \frac{3\ddot{f}_1(t_s)^2}{\dot{f}_1(t_s)^3} + \frac{\ddot{f}_1(t_s)^3}{f_1(t_s)\dot{f}_1(t_s)^4} + \frac{\ddot{f}_1(t_s)}{f_1(t_s)\dot{f}_1(t_s)^2} \right), & \text{when } \alpha > 4 \\ \frac{1}{4} \left(\frac{6\dot{f}_1(t_s)}{f_1(t_s)^2} + \frac{\ddot{f}_1(t_s)}{f_1(t_s)^3} \right) \left(3 + \frac{\dot{f}_1(t_s)}{f_1(t_s)^2} \right)^{-1} \frac{f_2(t_s)\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{f_1(t_s)\dot{f}_1(t_s)^2} (t_s-t)^{\alpha-4}, & \text{when } 4 > \alpha > 2 \\ \frac{1}{4} \left(3 + \frac{\dot{f}_1(t_s)}{f_1(t_s)^2} \right)^{-1} \frac{f_2(t_s)^2\alpha^2(\alpha-1)^2(\alpha-2)(\alpha-3)}{f_1(t_s)^4\dot{f}_1(t_s)^2} (t_s-t)^{2\alpha-6}, & \text{when } 2 > \alpha > 1 \end{cases}. \end{aligned} \tag{45}$$

If $f_1(t)$ is assumed to be smooth, the slow-roll parameter ϵ blows up when $2 > \alpha > 1$, which means that it develops a singularity, while it is regular for $\alpha > 2$. In addition, the slow-roll parameter η , blows up for $3 > \alpha > 1$, while ξ^2 blows up in two different ways when $2 > \alpha > 1$ and when $4 > \alpha > 2$. Therefore, when $\alpha > 4$, the slow-roll parameters contain no singularities at least in the vicinity

of the Type IV singularity. It is worth presenting some illustrative examples in order to further scrutinize the behavior of the slow-roll parameters. In the case that $H(t) = f_0(t - t_s)^\alpha$, the slow-roll parameter ϵ , becomes equal to,

$$\epsilon = \frac{f_0(t - t_s)^{-1+\alpha} \alpha (-1 + 6t - 6t_s + \alpha)^2}{4(3f_0(t - t_s)^{1+\alpha} + \alpha)^2}, \tag{46}$$

while the slow-roll parameter η is equal to,

$$\eta = \frac{(t - t_s)^{-3-\alpha} (-4t^2 + 8tt_s - 4t_s^2 + 4t^2\alpha - 8tt_s\alpha + 4t_s^2\alpha - t^2\alpha^2 + 2tt_s\alpha^2)}{4f_0(3f_0(t - t_s)^{1+\alpha} + \alpha)} + \frac{(t - t_s)^{-3-\alpha} (-t_s^2\alpha^2 + 2\alpha^3 - 2\alpha^4 - 6f_0(t - t_s)^{3+\alpha}(-1 + 3\alpha) + f_0^2(t - t_s)^{2\alpha}\alpha^2(1 + 2(-1 + \alpha)\alpha))}{4f_0(3f_0(t - t_s)^{1+\alpha} + \alpha)}. \tag{47}$$

Finally, the slow-roll parameter ξ^2 as a function of the cosmic time t , is equal to,

$$\xi^2 = \frac{(t - t_s)^{-5-2\alpha} (-1 + \alpha)(-1 + 6t - 6t_s + \alpha)}{4f_0^2(3f_0(t - t_s)^{1+\alpha} + \alpha)} \times \left(5(t - t_s)^2(-1 + \alpha)^2 + 3f_0^2(t - t_s)^{2\alpha}(-2 + \alpha)^2(-1 + \alpha)\alpha + 3f_0(t - t_s)^{3+\alpha}(1 + 2\alpha) \right). \tag{48}$$

By looking Eqs. (46), (47), and (48), we can easily see that the slow-roll parameters have singularities at the point $t = t_s$, where the Type IV singularity occurs, a fact that we also stressed in the general example we presented earlier. The singularity in the slow-roll parameters can be viewed as rather unwanted features of the theory, or these can indicate a strong instability of the dynamical system that describes the cosmological evolution. Work is in progress towards the latter possibility.

As a final task we shall try to compare the results of our analysis with the recent Planck data [21], by suitably choosing the phenomenological EoS function. For the purposes of our analysis, we shall express the observational indices as functions of the e -folding number N . We shall consider three important observational indices, namely the spectral index of primordial curvature perturbations n_s , the scalar-to-tensor ration r and the running of the spectral index α_s , which as was evinced in Ref. [6], these are written in terms of the e -folding number N as follows,

$$n_s - 1 = -9\rho(N)\tilde{f}(\rho(N)) \left(\frac{\tilde{f}'(\rho(N)) - 2}{2\rho(N) - \tilde{f}(\rho(N))} \right)^2 + \frac{6\rho(N)}{2\rho(N) - \tilde{f}(\rho(N))} \left\{ \frac{\tilde{f}(\rho(N))}{\rho(N)} + \frac{1}{2} (\tilde{f}'(\rho(N)))^2 + \tilde{f}'(\rho(N)) - \frac{5}{2} \frac{\tilde{f}(\rho(N))\tilde{f}'(\rho(N))}{\rho(N)} + \left(\frac{f(\rho)}{\rho(N)} \right)^2 + \frac{1}{3} \frac{\rho'(N)}{\tilde{f}(\rho(N))} \left[(\tilde{f}'(\rho(N)))^2 + \tilde{f}(\rho(N))\tilde{f}''(\rho(N)) - 2 \frac{\tilde{f}(\rho(N))\tilde{f}'(\rho(N))}{\rho(N)} + \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 \right] \right\}, \tag{49}$$

$$r = 24\rho(N)\tilde{f}(\rho(N)) \left(\frac{\tilde{f}'(\rho(N)) - 2}{2\rho(N) - \tilde{f}(\rho(N))} \right)^2, \tag{50}$$

$$\alpha_s = \rho(N)\tilde{f}(\rho(N)) \left(\frac{\tilde{f}'(\rho(N)) - 2}{2\rho(N) - \tilde{f}(\rho(N))} \right)^2 \times \left[\frac{72\rho(N)}{2\rho(N) - \tilde{f}(\rho(N))} J_1 - 54\rho(N)\tilde{f}(\rho(N)) \left(\frac{\tilde{f}'(\rho(N)) - 2}{2\rho(N) - \tilde{f}(\rho(N))} \right)^2 - \frac{1}{\tilde{f}'(\rho(N)) - 2} J_2 \right], \tag{51}$$

where the detailed functional form of J_1 and J_2 is given in Appendix A. In order to have a qualitative idea of the behavior of the observational indices in terms of the phenomenological EoS of Eq. (33), we shall study a not very sophisticated model, but simple nevertheless, that leads to a Type IV singularity. Suppose that the EoS is given by Eq. (33), with $f(\rho) = A\rho^\alpha$ and in order to proceed we have to express the function $f(\rho)$, in terms of the e -folding number N . This can be easily done, since the scale factor in this case is equal to,

$$a(t) = a_0 e^{\frac{\rho^{1-\alpha}}{3(1-\alpha)A}}, \tag{52}$$

so the effective energy density as a function of the e -folding number N is given by,

$$\rho = (3(1 - \alpha)A)^{\frac{1}{1-\alpha}} N^{\frac{1}{1-\alpha}}. \tag{53}$$

Recall that the function $f(\rho)$ can generate a Type IV singular evolution when $0 < \alpha < \frac{1}{2}$, as we demonstrated below Eq. (36) and therefore, the fraction $f(\rho)/\rho$ can be chosen to satisfy the constraint,

$$\frac{f(\rho)}{\rho} \ll 1. \tag{54}$$

In this case, the observational indices can be very much simplified, and can be approximated by the following expressions [6],

$$n_s \simeq 1 - 6 \frac{f(\rho)}{\rho(N)}, \quad r \simeq 24 \frac{f(\rho)}{\rho(N)}, \quad \alpha_s = -9 \left(\frac{f(\rho)}{\rho(N)} \right)^2. \quad (55)$$

Combining Eqs. (53) and (55), with the constraint (54) holding true, the resulting approximate expressions of the observational indices read,

$$n_s \simeq 1 - \frac{2}{N(1-\alpha)}, \quad r \simeq \frac{8}{N(1-\alpha)}, \quad \alpha_s \simeq -\frac{1}{N^2(1-\alpha)^2}. \quad (56)$$

The 2015 Planck report [21], restricts the values of the observational indices as follows,

$$n_s = 0.9644 \pm 0.0049, \quad r < 0.10, \quad \alpha_s = -0.0057 \pm 0.0071, \quad (57)$$

which can be relaxed if someone assumes a scale dependence of the scalar and tensor spectral indices. By using the values $(N, \alpha) = (60, 1/20)$, the observational indices become equal to,

$$n_s \simeq 0.96491, \quad r \simeq 0.1403, \quad \alpha_s = -0.000307. \quad (58)$$

Therefore, there is concordance with the spectral index of primordial curvature perturbations, but the scalar-to-tensor ratio and the associated running of the spectral index constraints are not satisfied. However, in principle concordance can be achieved, if a more sophisticated model is considered, instead of the simple model we used here, just for expositional purposes.

4. Conclusions

In this letter we investigated various phenomenological EoS models that can generate a Type IV singular cosmological evolution. The motivation for considering phenomenological EoS models is coming from the fact that the Universe's EoS seems to be non-constant and it may slightly cross the phantom divide at late-time. In addition, the Type IV singularity is the mildest possibility of singular evolutions, since it is of non-crushing type, and therefore the Hawking–Penrose theorems [22] are satisfied for this sort of singularity. However, as we evinced, the slow-roll parameters might become singular at the singularity points, for specific values of the free parameters that govern the cosmological evolution. The singularity of the slow-roll parameters might indicate some sort of instability of the dynamical system that the cosmological evolution equations constitute. This singularity of the slow-roll parameters was also observed in [16–18] when evolution is governed by scalars, so a repeating pattern seems to underlie this feature, which by no chance is accidental. This issue needs to be further scrutinized, and in general all the consequences of finite time singularities should be fully understood in the context of classical cosmology. This is owing to the fact that singularities, and especially the crushing singularities, appearing in classical theories are usually indicators of an underlying theoretical framework yet to be found. Therefore, the “mild” singularities might be some classical link between the quantum and the classical theory of gravity (apart from Type IV singularity where quantum effects should not play an essential role).

An interesting possibility that we did not address in this letter is to study a Type IV singular evolution caused by two distinct phenomenological EoS dark fluids, or even couple one fluid with dark matter. This study however exceeds the introductory purposes of this paper and is deferred to a future work.

Finally, a comment is in order: The generalized models we used are not by any means just mathematical constructions, since such EoS models are typical for acceleration of the Universe, since their EoS parameter is around the value -1 . Their specific form can be imposed in principle, by a specific evolution of the Universe, therefore these EoS are physically motivated by the early/current status of the evolution of the Universe.

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Appendix A

Here we provide the detailed functional form of the parameters J_1 and J_2 that appear in the expressions giving the observational indices, namely in Eq. (51), as functions of the effective energy density, and the e -folding number. The detailed form of these is,

$$J_1 \equiv \frac{\tilde{f}(\rho(N))}{\rho(N)} + \frac{1}{2} \left(\tilde{f}'(\rho(N)) \right)^2 + \tilde{f}''(\rho(N)) - \frac{5}{2} \frac{\tilde{f}(\rho(N)) \tilde{f}'(\rho(N))}{\rho(N)} + \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 + \frac{1}{3} \frac{\rho'(N)}{\tilde{f}(\rho(N))} \\ \times \left[\left(\tilde{f}'(\rho(N)) \right)^2 + \tilde{f}(\rho(N)) \tilde{f}''(\rho(N)) - 2 \frac{\tilde{f}(\rho(N)) \tilde{f}'(\rho(N))}{\rho(N)} + \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 \right], \quad (59)$$

$$J_2 \equiv \frac{45}{2} \frac{\tilde{f}(\rho(N))}{\rho(N)} \left(\tilde{f}'(\rho(N)) - \frac{1}{2} \frac{\tilde{f}(\rho(N))}{\rho(N)} \right) + 18 \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^{-1} \left\{ \left(\tilde{f}'(\rho(N)) - \frac{1}{2} \frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 \right.$$

$$\begin{aligned}
& + \left(\tilde{f}'(\rho(N)) - \frac{1}{2} \frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^3 \Bigg\} - 9 \left(\tilde{f}'(\rho(N)) - \frac{1}{2} \frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 - 45 \tilde{f}'(\rho(N)) + 9 \frac{\tilde{f}(\rho(N))}{\rho(N)} \\
& + 3 \left(4 \tilde{f}'(\rho(N)) - 7 \frac{\tilde{f}(\rho(N))}{\rho(N)} + 2 \right) \Bigg\{ - \frac{3}{2} \left(\tilde{f}'(\rho(N)) - \frac{1}{2} \frac{\tilde{f}(\rho(N))}{\rho(N)} \right) + \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^{-2} \frac{\rho'(N)}{\rho(N)} \\
& \times \left[\left(\tilde{f}'(\rho(N)) \right)^2 + \tilde{f}(\rho(N)) \tilde{f}''(\rho(N)) - 2 \frac{\tilde{f}(\rho(N)) \tilde{f}'(\rho(N))}{\rho(N)} + \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 \right] \Bigg\} \\
& + \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^{-2} \Bigg\{ - \frac{3}{2} \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right) \left(\frac{\rho'(N)}{\rho(N)} \right) \left[3 \left(\tilde{f}'(\rho(N)) \right)^2 + 2 \tilde{f}(\rho(N)) \tilde{f}''(\rho(N)) \right. \right. \\
& \left. \left. - \frac{11}{2} \frac{\tilde{f}(\rho(N)) \tilde{f}'(\rho(N))}{\rho(N)} + \frac{5}{2} \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 \right] \right. \\
& + \left(\frac{\rho''(N)}{\rho(N)} \right) \left[\left(\tilde{f}'(\rho(N)) \right)^2 + \tilde{f}(\rho(N)) \tilde{f}''(\rho(N)) - 2 \frac{\tilde{f}(\rho(N)) \tilde{f}'(\rho(N))}{\rho(N)} + \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 \right] \\
& + \left(\frac{\rho'(N)}{\rho(N)} \right)^2 \left[\left(3 \tilde{f}'(\rho(N)) \tilde{f}''(\rho(N)) + \tilde{f}(\rho(N)) \tilde{f}'''(\rho(N)) \right) \rho(N) - 3 \left(\tilde{f}'(\rho(N)) \right)^2 \right. \\
& \left. \left. - 3 \tilde{f}(\rho(N)) \tilde{f}''(\rho(N)) + 6 \frac{\tilde{f}(\rho(N)) \tilde{f}'(\rho(N))}{\rho(N)} - 3 \left(\frac{\tilde{f}(\rho(N))}{\rho(N)} \right)^2 \right] \right. \Bigg\}. \tag{60}
\end{aligned}$$

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