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Brief paper

Model predictive control for constrained systems with serially correlated stochastic parameters and portfolio optimization ☆

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ABSTRACT

In this paper, we consider MPC for constrained discrete-time systems with stochastic parameters which are assumed to be a set of serially correlated time series. A generalized performance criterion is composed of a weighted sum of a linear combination of the (a) expected value of quadratic forms of state and control vectors, (b) quadratic forms of the expected value of the state vector, and (c) the linear component of the expected value of the state vector. The purpose of the present paper is to design optimal control strategies that are independent of distributional assumptions on the stochastic parameters and subject to hard constraints on the input manipulated variables and to provide a numerically tractable algorithm for practical applications. All expressions are presented in terms of the first- and second-order conditional moments. The results are applied to a problem of investment portfolio optimization with serially correlated returns. We present the numerical modelling results, based on stocks traded on the Russian Stock Exchanges MICEX.

1. Introduction

Lately, there has been a steadily growing need and interest in systems with stochastic parameters and/or multiplicative noise. The same systems have been gaining greater acceptance in many engineering applications. Financial engineering is also an important field of application where such models are used for describing the evolution of investment portfolios (see, for instance, Costa and Araujo (2008), Dombrovskii and Lyashenko (2003), Hu and Zhou (2005)).

Several results related to control systems with stochastic parameters subject to constraints have already been derived. In recent years, considerable interest has been focused on model predictive control (MPC), also known as receding horizon control (RHC). MPC proved to be an appropriate and effective technique to solve the dynamic control problems subject to input and state/output constraints.

MPC for constrained discrete-time linear systems with random parameters and/or multiplicative noises has been intensively studied lately. Some of the recent works on this subject can be found, for instance, in Bemporad and Di Cairano (2011), Bernardini and Bemporad (2009), Calafiore and Fagiano (2013), Cannon, Kouvaritakis, and Wu (2009), Dombrovskii, Dombrovskii, and Lyashenko (2005, 2006), Dombrovskii and Obyedko (2011), Lee and Cooly (1998), and Primbs (2009).

In particular, Lee and Cooly (1998) investigate systems with independent and identically distributed parameters, while Dombrovskii et al. (2005) study systems with both control and state multiplicative noises and stochastic independent parameters under hard constraints on input variables. Cannon et al. (2009) study systems with control and state multiplicative noises where constraints are assumed to be soft and probabilistic. Primbs and Sung (2009) study systems with control and state multiplicative noises in the presence of soft quadratic expectation constraints. In Dombrovskii et al. (2006), MPC of linear systems with random dependent parameters, where the evolution of parameters is described by linear difference stochastic equations under hard constraints on the control variables, is considered. Dombrovskii and Obyedko (2011) investigate the MPC problem of discrete-time Markov jump linear systems with multiplicative noise subject to hard constraints on the control variables. Related results in MPC design via scenario generation can be found in Bemporad and Di Cairano (2011), Bernardini and Bemporad (2009), and Calafiore and Fagiano (2013). Note that the scenario-based MPC is often computationally demanding and assumes a specific probability distribution for the model parameters.

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In this paper, we consider MPC for constrained discrete-time systems with stochastic parameters which are assumed to be a set of serially correlated time series. The lead–lag relationships between component series are described by the matrices of the second-order conditional moments and the knowledge of the statistical distribution of the parameters is not assumed.

We consider a generalized performance criterion which is composed of a weighted sum of a linear combination of the (a) expected value of quadratic forms of state and control vectors, (b) quadratic forms of the expected value of the state vector, and (c) a linear part in the expected value of the state vector. The motivation for adopting this type of criterion is that in several situations we provide solutions for two special cases. The first one is MPC for the quadratic criterion and the second MPC for the mean–variance criterion. Note that this cost function is not traditionally used in MPC theory. This approach to cost function formulation is based on an idea proposed in Costa and Araujo (2008) that a generalized multi-period mean–variance portfolio selection problem is considered without constraints and with a finite horizon.

The main goal of the present paper is to design optimal control strategies that are independent of distributional assumptions on the stochastic parameters and are subject to hard constraints on the input manipulated variables and to provide a numerically tractable algorithm for practical applications. We derive an exact expression for the predicted performance criterion as an explicit function of predicted input variables that can be optimized online by minimizing over the vector of predicted input variables subject to hard constraints.

The results are applied to a problem of investment portfolio optimization with serially correlated returns. Note that the portfolio management problem is the key problem of financial engineering that includes a set of major problems associated with the control of complex dynamic systems with stochastic parameters under constraints. Therefore, the investment portfolio can be a powerful platform for testing the effectiveness of the designed control strategies.

There are many examples of the MPC in finance applications. Some recent works can be found in Bemporad, Puglia, and Gabbriellini (2011), Dombrovskii, Dombrovskii, and Lyashenko (2004), Dombrovskii et al. (2005, 2006), Dombrovskii and Obyedko (2011), Herzog, Dondi, and Geering (2007), and Primbs (2009). In all of these papers, authors assume the hypothesis of serially independent returns and/or consider the explicit form of the model describing the price process of the risky assets (e.g., geometric Brownian motion, etc.).

Related results in multi-period portfolio optimization can be found in Calafiore (2008, 2009) where a multi-stage optimization model is developed. In a developed model portfolio, diversity constraints are imposed in expectation (soft constraints). Calafiore (2008, 2009) in his works has proposed the use of a linearly parameterized class of feedback control policies that are affine functions of the past return innovations. However, it is difficult to obtain an exact analytic formulation of the optimization problem based on the proposed model for the case of generic and serially correlated return processes. Calafiore (2008, 2009) proposed an approximated technique to solve the problem via stochastic simulations of the return series that can be used in practice when a full stochastic model for return dynamics is available.

In this paper, we propose a framework for the computation of dynamic trading strategies subject to serially correlated returns and hard constraints on the trading amounts. The only conditions imposed on the distributions of the asset returns are the existences of the conditional mean vectors and of the conditional second-order moments. No assumptions about the correlation structure between different time points or about the distribution of the asset returns are needed. The proposed trading strategies are convenient to adaptive implementation. Adaptive algorithms have the ability to adapt to the underlying data by dynamically incorporating new information into the decision process and they are more suitable for non-stationary environments, such as those in finance. The motivation is within the context of algorithmic trading, which demands fast and recursive updates of portfolio allocations as new data arrives.

We want to demonstrate the performance of our model under real market conditions. We present the numerical modelling results, based on stocks traded on the Russian Stock Exchanges MICEX.

2. Problem formulation

We consider the following discrete-time system with stochastic parameters on the probabilistic space $(\Omega, \mathfrak{F}, \mathbf{P})$.

$$x(k+1) = Ax(k) + B[\eta(k+1), k+1]u(k), \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the vector of state, $u(k) \in \mathbb{R}^{n_u}$ is the vector of control inputs, and $\eta(k) \in \mathbb{R}^q$ is assumed to be a stochastic time series. The matrices $A \in \mathbb{R}^{n_x \times n_x}, B[\eta(k), k] \in \mathbb{R}^{n_x \times n_u}$ are the system matrix and the input matrix, respectively. All of the elements of $B[\eta(k), k]$ are assumed to be linear functions of $\eta(k)$.

Let $\mathbb{F} = (\mathfrak{F}_k)_{k \geq 1}$ be the complete filtration with σ-field $\mathfrak{F}_k$ generated by the $\{\eta(s) : s = 0, 1, 2, \ldots, k\}$ that models the flow of information to time k.

Throughout the paper, we use the following notations. We denote with $E\{a/b\}$ the conditional expectation of a with respect to b. For any matrix $\psi[\eta(k+i), k+i]$, dependent on $\eta(k+i)$, $\overline{\psi}(k+i) = E\{\psi[\eta(k+i), k+i]/\mathfrak{F}_k\}$, $\tilde{\psi}(k+i) = \psi(k+i) - \overline{\psi}(k+i)$, $i \geq 1$, without indicating the explicit dependence of matrices on $\eta(k+i)$. Additionally, we use the standard notation, for square matrix M, $M \geq 0 (M > 0)$ to denote that the matrix M is positive semidefinite (positive definite).

We allow the time series $\eta(k)$ to be serially correlated. Let us assume that we know the first- and second-order conditional moments for the stochastic vector $\eta(k)$ about $\mathfrak{F}_k$:

$$E\{\eta(k+i)/\mathfrak{F}_k\} = \overline{\eta}(k+i),$$
$$E\left\{\eta(k+i)\eta^T(k+j)/\mathfrak{F}_k\right\} = \Theta_{ij}(k), \quad (i, j = 1, 2, \ldots, l).$$

Therefore, the lead–lag relationships between component series $\eta_t(k+i)$ and $\eta_f(k+j)$ are described by the matrices $\Theta_{ij}(k)$ of the second-order conditional moments.

We impose the following inequality constraints on the control inputs (element-wise inequality)

$$u_{\min}(k) \leq S(k)u(k) \leq u_{\max}(k), \tag{2}$$

where $S(k) \in \mathbb{R}^{p \times n_u}$; $u_{\min}(k), u_{\max}(k) \in \mathbb{R}^p$.

We use the MPC methodology in order to define the optimal control strategy. The main concept of MPC is to solve an open-loop constrained optimization problem at each time instant and to implement only the initial optimizing control action of the solution.

We define the following cost function with receding horizon, which is to be minimized at every time k,

$$\begin{aligned} J(k+m/k) &= \sum_{i=1}^{m} E\{x^T(k+i)R_1(k+i)x(k+i)/x(k), \mathfrak{F}_k\} \\ &- \sum_{i=1}^{m} E\{x^T(k+i)/x(k), \mathfrak{F}_k\}R_2(k+i)E\{x(k+i)/x(k), \mathfrak{F}_k\} \\ &- \sum_{i=1}^{m} R_3(k+i)E\{x(k+i)/x(k), \mathfrak{F}_k\} \\ &+ \sum_{i=0}^{m-1} E\{u^T(k+i/k)R(k+i)u(k+i/k)/x(k), \mathfrak{F}_k\}, \end{aligned} \tag{3}$$

on trajectories of system (1) over the sequence of predictive control inputs $u(k/k), \ldots, u(k+m-1/k)$ dependent on information up to time k, under constraints (2), where $R_1(k+i) \geq 0, R_2(k+i) \geq 0$, and $R(k+i) > 0$ are given symmetric weight matrices of corresponding dimensions, $R_3(k+i)$ is given vector; m is the prediction horizon.

Only the first control vector $u(k/k)$ is actually used for control. Thereby we obtain control $u(k)$ as a function of $\mathfrak{F}_k$ and $x(k)$, i.e., the feedback control. This optimization process is solved again at the next time instant $k+1$ to obtain control $u(k+1)$.

Different cost functions can be obtained from this criterion after setting the coefficients $R_1(k+i)$, $R_2(k+i)$, and $R_3(k+i)$ to some appropriate values.

Problem 2.1. Taking $R_2(k+i) = 0$, we have the MPC problem with quadratic criterion, composed by a linear combination of quadratic and linear parts.

Problem 2.2. Let system (1) have a scalar output $y(k) = L(k)x(k)$, where $L(k)$ is a vector of appropriate dimension. Taking

$$R_1(k+i) = R_2(k+i) = \mu(k+i)L^T(k+i)L(k+i),$$
$$R_3(k+i) = \rho(k+i)L(k+i), \quad (i = \overline{1,m}),$$

where $\mu(k+i) \geq 0$, $\rho(k+i) \geq 0$ are scalar values, we have a mean–variance optimization problem. The input parameters $\mu(k+i)$, and $\rho(k+i)$ can be seen as risk aversion coefficients, giving a trade-off between the expected system state and the associated risk (variance) level at time k.

3. Model predictive control strategy design

Consider the problem of minimizing the objective (3) with respect to the predictive control variables $u(k+i/k), i = 0, 1 \ldots, m-1$, subject to constraints (2).

Theorem 3.1. *Let the system dynamics be given by (1) under constraints (2). Then, the MPC policy with receding horizon m, such that it minimizes the objective (3), for each instant k is defined by*

$$u(k) = \begin{bmatrix} I_{n_u} & 0_{n_u} & \ldots & 0_{n_u} \end{bmatrix} U(k),$$

where I_{n_u} is a n_u-dimensional identity matrix, 0_{n_u} is a n_u-dimensional zero matrix, and $U(k) = [u^T(k/k), \ldots, u^T(k+m-1/k)]^T$ is the set of predictive controls defined from the solving of the quadratic programming problem with criterion

$$Y(k+m/k) = \left[2x^T(k)G(k) - F(k)\right]U(k) + U^T(k)H(k)U(k) \quad (4)$$

under constraints

$$U_{\min}(k) \leq \overline{S}(k)U(k) \leq U_{\max}(k), \quad (5)$$

where

$$\overline{S}(k) = diag(S(k), \ldots, S(k+m-1)),$$
$$U_{\min}(k) = [u^T_{\min}(k), \ldots, u^T_{\min}(k+m-1)]^T,$$
$$U_{\max}(k) = [u^T_{\max}(k), \ldots, u^T_{\max}(k+m-1)]^T, \quad and$$

$H(k), G(k), F(k)$ are the block matrices and blocks satisfy the following equations

$$\begin{aligned} H_{tt}(k) = {} & R(k+t-1) + E\{B^T[\eta(k+t), k+t] \\ & \times [Q_1(m-t) - Q_2(m-t)]B[\eta(k+t), k+t]/\mathfrak{F}_k\} \\ & + E\{\widetilde{B}^T(k+t)Q_2(m-t)\widetilde{B}(k+t)/\mathfrak{F}_k\}, \end{aligned} \quad (6)$$

$$\begin{aligned} H_{tf}(k) = {} & E\{B^T[\eta(k+t), k+t](A^T)^{f-t} \\ & \times [Q_1(m-f) - Q_2(m-f)]B[\eta(k+f), k+f]/\mathfrak{F}_k\} \\ & + E\{\widetilde{B}^T(k+t)(A^T)^{f-t}Q_2(m-f)\widetilde{B}(k+f)/\mathfrak{F}_k\}, \quad t < f, \end{aligned} \quad (7)$$

$$H_{tf}(k) = H^T_{ft}(k), \quad t > f, \ (t, f = \overline{1,m}),$$

$$G_t(k) = (A^t)^T[Q_1(m-t) - Q_2(m-t)]\overline{B}(k+t), \quad (8)$$

$$F_t(k) = Q_3(m-t)\overline{B}(k+t), \quad (9)$$

$$Q_1(t) = A^T Q_1(t-1)A + R_1(k+m-t),$$
$$Q_2(t) = A^T Q_2(t-1)A + R_2(k+m-t),$$
$$Q_3(t) = Q_3(t-1)A + R_3(k+m-t),$$
$$Q_1(0) = R_1(k+m), \qquad Q_2(0) = R_2(k+m),$$
$$Q_3(0) = R_3(k+m).$$

If $R_2(k+i) = 0$, $(i = \overline{1,m})$ then $Q_2(i) = 0$ and we have optimal control strategies based on quadratic criterion (Problem 2.1).

If $R_1(k+i) = R_2(k+i) = \mu(k+i)L^T(k+i)L(k+i)$, $R_3(k+i) = \rho(k+i)L(k+i)$, $(i = \overline{1,m})$, then $Q_1(i) = Q_2(i)$, $G(k) = 0$ and we have optimal control strategies based on mean–variance criterion (Problem 2.2).

A proof of this theorem is reported in Appendix A.

Remark 3.1. Note that the optimal controller does not assume any specific probability distribution for the model parameters. The solution exclusively depends on the conditional mean vectors and the conditional second-order moments matrices. These quantities can be directly obtained by applying the well-developed theory of multivariate time series (Lütkepohl, 2005; Tsay, 2002).

Remark 3.2. The condition $R(k+i) > 0$ guarantees that the matrix $H(k)$ is positive definite for two particular problems: Problems 2.1 and 2.2. Thus in both cases the solution of a quadratic programming task with a criterion (4) exists and is unique if the constraints (5) are admissible.

Remark 3.3. It might seem computationally difficult to calculate the conditional moments in expressions (6) and (7). However, due to the assumption of linear dependence of matrix $B[\eta(k+i), k+i]$, $(i = \overline{1,m})$ on $\eta(k+i)$, these values depend on conditional moments and can be easily calculated because we assume that conditional moments are known.

4. Portfolio optimization problem

In this section, we present an application of the previous results to a portfolio optimization problem. The portfolio management question addresses choosing how to allocate money into different securities with some objective defined by the investor. We use the same portfolio model as in Dombrovskii et al. (2006). We introduce two dynamic portfolio optimization problems. Each of the problems is solved by using the algorithm given by Theorem 3.1.

Consider an investment portfolio consisting of n risky assets and one risk-free asset (e.g., a bank account or a government bond). We assume that the decision time horizon is composed of a large number of periods. During each such period, the decision-maker (investor) obtains new information about the returns (prices of assets) and reacts to a new market situation (information) by selling some assets and acquiring some other assets.

Let $u_i(k)$ $(i = 1, 2, \ldots, n)$ denote the amount of the wealth invested in the ith risky asset at time k; $u_0(k)$ is the amount invested in the risk-free asset. Then, the wealth process $V(k)$ satisfies

$$V(k) = \sum_{i=1}^{n} u_i(k) + u_0(k). \quad (10)$$

Let $\eta_i(k+1)$ denote the (simple) return of the ith risky asset per period $[k, k+1]$. It is a stochastic unobservable at time k with the value defined as

$$\eta_i(k+1) = \frac{P_i(k+1) - P_i(k)}{P_i(k)},$$

where $P_i(k)$ denotes the market value of the ith risky asset at time k.

By considering self-finance strategies (self-financing means that we do not allow wealth to be added to or extracted from the portfolio), the wealth process at the time $k+1$ is given by

$$V(k+1) = \sum_{i=1}^{n} [1 + \eta_i(k+1)]\, u_i(k) + [1+r]\, u_0(k), \tag{11}$$

where r is a risk-free interest rate.

From (10), we have $u_0(k) = V(k) - \sum_{i=1}^{n} u_i(k)$. Then we can rewrite (11) as follows:

$$V(k+1) = [1+r]\, V(k) + b[\eta(k+1), k+1]u(k),$$

where $\eta(k) = [\eta_1(k)\eta_2(k)\dots\eta_n(k)]^T$ is the vector of risky asset returns, $u(k) = [u_1(k) \quad u_2(k) \quad \dots \quad u_n(k)]^T$ is the vector of input (manipulated) variables, and

$$b\,[\eta(k), k] = \begin{bmatrix}\eta_1(k) - r & \eta_2(k) - r & \dots & \eta_n(k) - r\end{bmatrix}.$$

We impose the following constraints on the manipulated variables (trading amounts):

$$u_i^{\min}(k) \le u_i(k) \le u_i^{\max}(k), \quad (i = \overline{1, n}), \tag{12}$$

$$u_0^{\min}(k) \le V(k) - \sum_{i=1}^{n} u_i(k) \le u_0^{\max}(k). \tag{13}$$

If $u_i^{\min}(k) < 0$, $(i = 1, 2, \dots, n)$, we suppose that the amounts of the short-sale are restricted by $|u_i^{\min}(k)|$; if short-selling is prohibited, then $u_i^{\min}(k) \ge 0$, $(i = 1, 2, \dots, n)$. The amounts of long-sale are restricted by $u_i^{\max}(k)$, $(i = 1, 2, \dots, n)$. If $u_0^{\min}(k) < 0$, we can borrow capital (risk-free assets), then the volume of borrowing is restricted by $|u_0^{\min}(k)|$; $u_0^{\max}(k) \ge 0$ defines the upper bound of the amount we can invest in the risk-free asset. Note, that values $u_i^{\min}(k)$ and $u_i^{\max}(k)$ $(i = 0, 1, \dots, n)$ are often dependent on the common wealth of the portfolio in practice. So, we can write $u_i^{\min}(k) = \beta_i V(k)$ and $u_i^{\max}(k) = \gamma_i V(k)$, where β_i and γ_i are constant parameters.

Constraints (12)–(13) can be rewritten in matrix form (elementwise inequality):

$$u_{\min}(k) \le S u(k) \le u_{\max}(k), \tag{14}$$

where $S = \begin{bmatrix} I_n & -E^T \end{bmatrix}^T$, $E = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$,

$$u_{\min}(k) = \begin{bmatrix} u_1^{\min}(k) & u_2^{\min}(k) & \dots & u_n^{\min}(k) & u_0^{\min}(k) - V(k) \end{bmatrix}^T,$$

$$u_{\max}(k) = \begin{bmatrix} u_1^{\max}(k) & u_2^{\max}(k) & \dots & u_n^{\max}(k) & u_0^{\max}(k) - V(k) \end{bmatrix}^T.$$

Let us assume that the vectors of risky asset returns $\eta(k) = [\eta_1(k)\eta_2(k)\dots\eta_n(k)]^T$, $k = 0, 1, \dots,$ form a serially correlated non-stationary discrete-time multivariate process with finite conditional moments

$$E\{\eta(k+i)/\mathfrak{F}_k\} = \overline{\eta}(k+i),$$

$$E\left\{\eta(k+i)\eta^T(k+j)/\mathfrak{F}_k\right\} = \Theta_{ij}(k), \quad (i, j = \overline{1, l};\ k = 0, 1, 2, \dots).$$

One motivation for such a model is the fact that a large number of empirical analyses of assets' price dynamics show that there exists salient serial correlations in the returns of financial assets (Fama & French, 1988; Tsay, 2002).

We use the MPC methodology in order to define the optimal control portfolio strategy. The advantage of using a receding horizon implementation is that at each decision stage, we can profit from observations of actual market behaviour during the preceding period and use this information to feed fresh estimates to the model.

We define two portfolio control problems.

Problem 4.1. Our objective is to control the investment portfolio, via dynamic asset allocation among the n stocks and a risk-free asset, by tracking, as closely as possible, a desired deterministic reference trajectory

$$V^0(k+1) = [1 + \mu_0]V^0(k), \tag{15}$$

where μ_0 is a given parameter representing the growth factor and the initial state is $V^0(0) = V(0)$.

So, we have a dynamic tracking problem of a reference portfolio (15) with desired return μ_0 subject to constraints (14) with criterion

$$\begin{aligned} J(k+m/k) = E \Bigg\{ &\sum_{i=1}^{m} \left[V(k+i) - V^0(k+i)\right]^2 \\ &- \rho(k+i)\left[V(k+i) - V^0(k+i)\right] \\ &+ u^T(k+i-1/k)R(k, i-1)u(k+i-1/k)/V(k), \mathfrak{F}_k \Bigg\}, \end{aligned} \tag{16}$$

where m is the prediction horizon, $u(k+i/k) = [u_1(k+i/k), \dots, u_n(k+i/k)]^T$ is the predictive control vector, $R(k+i) > 0$ is a positive symmetric matrix of control cost coefficients, and $\rho(k+i) \ge 0$ is the weight coefficient.

The performance criterion (16) is composed by a linear combination of a quadratic part, representing the conditional mean-square error between the investment portfolio value and a reference (benchmark) portfolio, and a linear part, penalizing wealth values that are less than the desired value. The trade-off between these two terms is balanced by the weight $\rho(k+i)$.

Criterion (16) can be transformed into the equivalence form

$$\begin{aligned} J(k+m/k) = \sum_{i=1}^{m} \Big[&E\left\{V^2(k+i)/V(k), \mathfrak{F}_k\right\} \\ &- \left[2V^0(k+i) + \rho(k+i)\right] E\{V(k+i)/V(k), \mathfrak{F}_k\} \\ &+ E\{u^T(k+i-1/k)R(k, i-1)u(k+i-1/k)/V(k), \mathfrak{F}_k\} \Big], \end{aligned}$$

where we eliminated the term that is independent on control variables.

Problem 4.2. We define a multi-period mean–variance portfolio optimization problem with criterion

$$\begin{aligned} J(k+m/k) = \sum_{i=1}^{m} [E\{(V(k+i) \\ &- E\{V(k+i)/V(k), \mathfrak{F}_k\})^2 /V(k), \mathfrak{F}_k\} \\ &- \rho(k+i)E\{V(k+i)/V(k), \mathfrak{F}_k\} \\ &+ E\{u^T(k+i-1/k)R(k+i-1)u(k+i-1/k)/V(k), \mathfrak{F}_k\}], \end{aligned}$$

where the input parameter $\rho(k+i)$ denotes the level of risk aversion, giving a trade-off between the expected portfolio value and the associated risk (variance) level at time k.

It is obvious that the results presented in Theorem 3.1 can be applied to solve Problems 4.1 and 4.2.

5. A real data numerical example

The purpose of this chapter is to demonstrate the efficiency and the powerful practical potential of the proposed algorithms using the example of stochastic system as complex as an investment portfolio. In this section, our approach is tested on a set of real stocks. The data used for these tests are taken from the Russian Stock Exchange MICEX (www.finam.ru). They include the actual daily closing stock prices of the largest companies such as Sberbank, Gazprom, VTB, LUKOIL, NorNickel, Rosneft, and Sibneft. The portfolio was composed of five risky assets. Performing numerical modelling, we looked over all of the possible combinations of the five assets.

We consider the situation of an investor who has to allocate one unit of wealth over the investment horizon of approximately 1000 trading days (about four years) among risky assets and one risk-free asset. The risk-free asset considered here is a bank account with risk-free rate $r_1 = 0.00005$ per day (approximately 2% per annum). The updating of the portfolio based on the MPC is executed once every trading day. For our portfolio, we assumed an initial wealth of $V(0) = V^0(0) = 1$. The weight coefficients are set as $R = diag(10^{-4}, \ldots, 10^{-4})$. We impose hard constraints on the portfolio problem with parameters $\beta_i = -0.6$, $\gamma_i = 3$ $(i = 1, \ldots, n)$, and $\gamma_0 = 3$. In this example, short-selling is allowed. For the case of tracking a reference portfolio (Strategy 1), we set the tracking target to return 0.15% per day ($\mu_0 = 0.0015$) and the weight $\rho(k+i) = 0$. For the case of mean–variance approach (Strategy 2), we set the weight as (1) $\rho(k+i) = 0.005$ and (2) $\rho(k+i) = 0.002$. For the on-line finite horizon MPC problems, we used a horizon of $m = 10$, and numerically solved it in MATLAB by using the quadprog.m function.

At each time k, the optimization problem requires as input parameters the predicted returns and predicted second moments of returns over the predictive horizon m. These parameters can be estimated using different model specifications describing the return asset evolution. Examples include using autoregressive models, conditional heteroscedastic models, factor models, complex non-parametric methods and others (see, for instance Chan, Karceski, & Lakonishok, 1999; Lütkepohl, 2005; Tsay, 2002).

As a simple example, we assume that the multivariate process of risky asset returns follows the VAR(1) model (vector autoregressive model of order 1) (Lütkepohl, 2005; Tsay, 2002)

$$\eta(k+1) = \nu + A_1\eta(k) + \omega(k+1),$$

where A_1 is a coefficient matrix, $\nu = (I_n - A_1)\mu$ is a vector of intercept terms, $\mu = E\{\eta(k)\}$; and $\omega(k+1)$ is an n-dimensional white noise, that is, $E\{\omega(k+1)\} = 0$; $E\{\omega(k+1)\omega^T(k+1)\} = \sigma$; $E\{\omega(k+i)\omega^T(k+j)\} = 0, i \neq j$.

The covariance matrix σ is assumed to be nonsingular.

We estimated parameters of this model by the ordinary least squares method using the observed historical data based on the past 200 trading days prior to the tracking period. These parameters were considered constant along the entire period under study and equal to the initial empirical estimates, based on backwards data. We calculated the predicted conditional second moments based on this VAR(1) model and substituted them into Eqs. (6)–(7).

In practice, time series of risky asset returns have a trending behaviour which is not compatible with the assumptions of the classical VAR model. In order to capture short-run trends of risky asset returns, we use the following modification of the forecasting procedure based on the VAR(1) model. We calculate the sample means of returns $\hat{\eta}(k)$ using 10-day windows of past historical return data and incorporate these estimates in the VAR(1) - predictor $\hat{E}\{\eta(k+h)/\eta(k)\} = (I_n + \hat{A}_1 + \hat{A}_1^2 + \cdots + \hat{A}_1^{h-1})\hat{\nu}(k) + \hat{A}_1^h\eta(k)$, where the true coefficients ν, A_1 are replaced by estimators $\hat{\nu}, \hat{A}_1$; $\hat{\nu} = (I_n - \hat{A}_1)\hat{\eta}(k)$; $h = 1, 2, \ldots, m$. This predictor is used to predict the expected returns over the predictive horizon m at each decision time k in Eqs. (8) and (9). When a new measurement becomes available, the oldest measurement is discarded and the new measurement is added. So, we use the adjusted procedure, updating the estimates of mean returns at each time k. One motivation for such a heuristic approach is that we have no restrictions to construct any type of predictors in order to obtain the best asset allocation strategies.

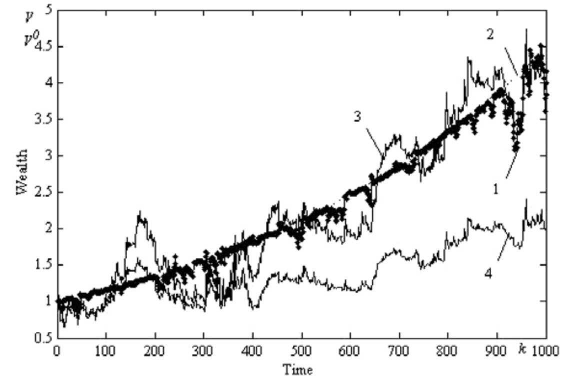

Fig. 1. Line 1 — tracking portfolio values (Strategy 1); 2— reference portfolio values; 3 — mean–variance portfolio values with $\rho = 0.005$ (Strategy 2); 4 — mean–variance portfolio values with $\rho = 0.002$ (Strategy 2).

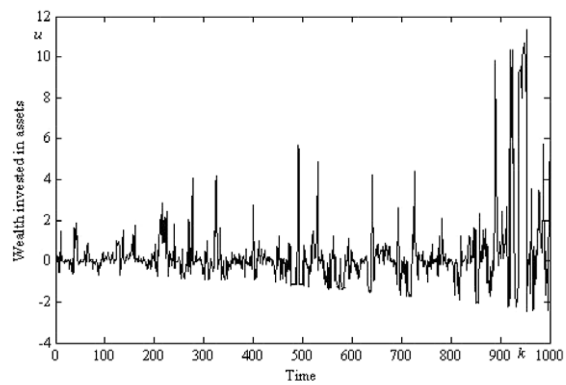

Fig. 2. Wealth invested in NorNickel for tracking a reference portfolio approach.

We present the typical results of the experiments on Figs. 1–4. In the pictures below, the portfolio was composed of five risky assets: LUKOIL, Gazprom, Sberbank, Rosneftj, and NorNickel. Investment period is from 17.06.2010 to 10.06.2014 (approximately 4 years). Fig. 1 plots the tracking portfolio (line 1) and a reference portfolio (line 2) values for Strategy 1 and a mean–variance performance for Strategy 2 with $\rho(k+i) = 0.005$ (line 3) and $\rho(k+i) = 0.002$ (line 4). Fig. 1 (see, Fig. 1, lines 3 and 4) shows that when the risk is greater, the yield is higher and vice versa. In Figs. 2 and 3, we have investments in the risky asset NorNickel for two control strategies. Fig. 4 plots risky asset returns for asset NorNickel.

Several insights can be gathered from the examples illustrated above. Fig. 1 shows that the tracking a reference portfolio strategy allows us to obtain a smoother curve of growth compared with the mean–variance strategy. The advantage of the control according to the quadratic criterion is that it is possible to predict the trajectory of the growth of portfolio wealth, which should follow as close as possible to the deterministic benchmark given by the investor.

It is important to acknowledge that in our experiments, where we use a rather simple model for parameters estimation, the performance of proposed strategies appears to be rather efficient. So, our approach allows us to design strategies which are desensitized, i.e., robustified, to parameters estimation. It is clear that one can

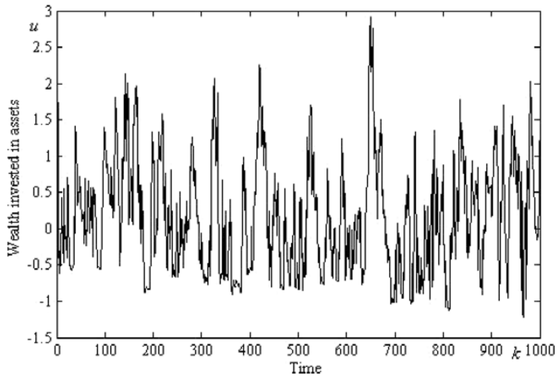

Fig. 3. Wealth invested in NorNickel for mean–variance approach, $\rho = 0.005$.

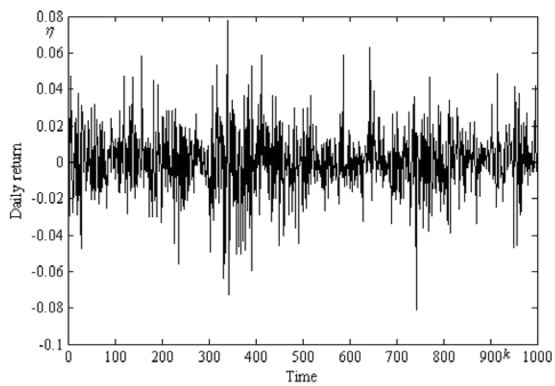

Fig. 4. Risky asset returns (NorNickel).

use more sophisticated estimation schemes to improve the precision of parameters estimation.

6. Conclusion

In this paper, we have offered a predictive control strategy for discrete-time systems with stochastic parameter uncertainties subject to hard constraints on input variables. System parameters are assumed to be correlated vector sequences for which only the first and the second conditional moments are known. The knowledge of the statistical distributions of the parameters is not assumed. We consider the control problem with a receding horizon under a generalized criterion. The proposed approach was applied to the control of stochastic system as complex as an investment portfolio.

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Appendix

Proof of Theorem 3.1. Let us introduce the following Lyapunov-type equations

$$Q_1(t) = A^T Q_1(t-1)A + R_1(k+m-t), \quad (t = \overline{1,m}), \tag{17}$$

$$Q_2(t) = A^T Q_2(t-1)A + R_2(k+m-t), \quad (t = \overline{1,m}), \tag{18}$$

and equation

$$Q_3(t) = Q_3(t-1)A + R_3(k+m-t), \quad (t = \overline{1,m}), \tag{19}$$

starting with $Q_1(0) = R_1(k+m)$, $Q_2(0) = R_2(k+m)$, $Q_3(0) = R_3(k+m)$. Straightforward calculations lead to the following expression for $J(k+m/k)$:

$$\begin{aligned}
J(k+m/k) &= x^T(k)A^T[Q_1(m-1) - Q_2(m-1)]Ax(k) \\
&+ 2x^T(k)\sum_{i=1}^{m}(A^i)^T[Q_1(m-i) - Q_2(m-i)] \\
&\times \bar{B}(k+i)u(k+i-1/k) \\
&+ \sum_{i=1}^{m} u^T(k+i-1/k)\left\{E\{B^T[k+i,\eta(k+i)]\right. \\
&\times [Q_1(m-i) - Q_2(m-i)]B[k+i,\eta(k+i)]/\mathfrak{F}_k\} \\
&+ E\{\widetilde{B}^T(k+i)Q_2(m-i)\widetilde{B}(k+i)/\mathfrak{F}_k\} \\
&+ \left. R(k+i-1)\right\} u(k+i-1/k) \\
&+ 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} u^T(k+i-1/k)\left\{E\{B^T[k+i,\eta(k+i)](A^T)^{j-i}\right. \\
&\times Q_1(m-j)B[k+j,\eta(k+j)]/\mathfrak{F}_k\} \\
&+ \left. E\{\widetilde{B}^T(k+i)(A^T)^{j-i}Q_2(m-j)\,\widetilde{B}(k+j)/\mathfrak{F}_k\}\right\} u(k+j-1/k) \\
&- \sum_{i=1}^{m} Q_3(m-i)\bar{B}(k+i)u(k+i-1/k) - Q_3(m-1)Ax(k),
\end{aligned} \tag{20}$$

where $Q_1(i)$, $Q_2(i)$, $Q_3(i)$ are defined by Eqs. (17)–(19).

Criterion (20) can be written in matrix form

$$\begin{aligned}
J(k+m/k) &= x^T(k)A^T[Q_1(m-1) - Q_2(m-1)]Ax(k) \\
&- Q_3(m-1)Ax(k) + \left[2x^T(k)G(k) - F(k)\right]U(k) \\
&+ U^T(k)H(k)U(k).
\end{aligned} \tag{21}$$

It is obvious that the problem of minimizing the criterion (21) is equivalent to the problem of minimizing the criterion (4). Thus, we have that the problem of minimizing the criterion (3) subject to (2) is equivalent to the quadratic programming problem with criterion (4) subject to (5). This completes the proof.

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