Алгебра и анализ
Том 24 (2012), № 4

St. Petersburg Math. J.
Vol. 24 (2013), No. 4, Pages 555–574
S 1061-0022(2013)01253-2
Article electronically published on May 24, 2013

# ELLIPTIC SOLITONS, FUCHSIAN EQUATIONS, AND ALGORITHMS

YU. V. BREZHNEV

Abstract. It is shown how the elliptic finite-gap potentials of the Schrödinger equation give rise to a family of solvable linear differential equations of the Fuchs class on the plane and on the torus: the latter case cannot be integrated via realizations of the Zinger–Kovacic type algorithms known in the Picard–Vessiot theory. For the arising Fuchsian equations, monodromy groups and their representations are constructed, the differential Galois group is described, together with a (recursive) method for calculation of the objects involved therein.

## §1. Introduction

Relatively recently, V. Kuznetsov (private communication, 2003) revealed a relationship between the theory of exact integration of the spectral problem

$$\Psi'' = \{\phi(x) + \lambda\}\Psi \tag{1}$$

(a part of the extensive theory of finite-gap integration) and the famous algorithmic method of Kovacic [31]. Historically, the latter is the first algorithm (it appeared in the late 1970s in the form of a technical report/preprint) for finding the solutions of a second order linear differential equation that are Liouville over $\mathbb{C}(x)$. Subsequently, some other integration methods were developing intensely in the early 1980s; they were applicable also to higher order equations, see [36]. Often, such algorithms are based on the property of equations to be Fuchsian.

On the other hand, finite-gap methods deliver numerous integrable families in which the solutions and potentials $u = \phi(x)$ are expressed in terms of theta functions. These functions are transcendental, i.e., irrational, like in the algorithms mentioned; apparently, this is the reason for which, until recently, $\Theta$-functional and related methods have been passing unnoticed by the differential Galois theory (the theory of Picard–Vessiot). Somewhat unexpectedly, the attributes of finite-gap integration do not occur in the main body of literature on the Galois theory [35, 2, 30], and *vice versa*, though both research fields deal with exact integration of the same equations. Even references to one of these fields within the scope of the other pertain to special cases [34, 2]. Extensive references can be found, e.g., in the books [35, 2]. However, a close relationship between the two theories was discovered long ago [24], which was emphasized in [20]. It should be noted that, recently, the supersymmetric quantum mechanics [14] was naturally embedded in the Galois theory, including the Kovacic algorithm, and that the "Hamiltonian algebraization" procedure leads to potentials represented by not only rational but also transcendental functions, see [15, §6.2].

---

2010 *Mathematics Subject Classification.* Primary 35C08; Secondary 35J10.

*Key words and phrases.* Elliptic solitons, Fuchsian equations, monodromy groups, integration methods, Kovacic algorithm.

Supported by the Federal Program "Scientific and pedogogical personnel of innovative Russia", grant no. 14.B37.21.0911.

Since any second order linear equation can be reshaped by a point transformation into any other equation of this sort, transition from one theory to the other is done via some transcendental substitutions. The first examples of such substitutions were found by Hermite and Darboux [22] long before the origin of the theories mentioned. This is the change $x \mapsto z = \wp(x)$, which takes the $n$-gap Lamé potentials $\phi(x) = n(n+1)\wp(x)$ to their algebraic form [12, §23.4]. By the same change, generalizations of such potentials become differential equations with rational coefficients [22]. As objects of the finite-gap theory, the potentials themselves (more precisely, their even elliptic representatives) were "transformed" into rational potentials relatively recently in the paper [38]. There, a Fuchsian equation with four singularities (Heun's equation) was treated, and in the subsequent paper [39] we see equations with a larger number of singularities. Also in [39], there are examples of explicit solutions and other results, though with no reference to algorithms and Galois theory.

It should be mentioned that for either of the two approaches there are cases where the other approach does not work. Thus, combining these two views may be of value for both. The first has a formal algorithmic nature and was automatized long ago, white the second is fairly general relative to integrability in broad context.

In all the examples mentioned above, we deal with linear equations of the Fuchs class. They are of great importance, and, by definition [10], they are not restricted to equations $\Psi'' = Q(x)\Psi$ with rational coefficients. The coefficients may be arbitrary (algebraic or transcendental) functions, and the specific class is only determined by the condition that for any point $x = \varrho$ of the (compact) manifold where the equation in question is defined, we can always find a solution of the form

$$\begin{aligned} \Psi &= (x-\varrho)^{\varkappa} \cdot \textit{a holomorphic function} \\ &= (x-\varrho)^{\varkappa} \cdot \big\{1 + a_1(x-\varrho) + \dots\big\} \end{aligned} \tag{2}$$

with some $\varkappa \in \mathbb{C}$. Such equations give a natural way to describe Riemann surfaces, their moduli spaces, isomonodromic deformations on algebraic curves, as well as extensive applications of these objects. Recently, an attempt has been made to establish a direct relationship of these objects with the finite-gap theory of the 1-gap Lamé equation. Though the equations arose long ago, their use is impeded by the poor elaboration of the theory in the case of irrational coefficients. Some rare and isolated examples are scattered in the literature (see, e.g., [25, Chapter X]), but no equations with natural origin are known. In this paper, we demonstrate such families that come from finite-gap potentials expressed in terms of elliptic functions. Note that already the elliptic finite-gap equations (1) themselves yield a solvable class for an algebraic manifold of genus 1, and consideration of the general class of such equations dates back to Klein [29, pp. 57–81].

Not only equations on elliptic curves arise in connection with finite-gap potentials, but also, as a particular case, some rational equations, namely, the Fuchsian equations treated in [38, 39]. In other words, we extend and naturally complete the investigations of [38, 39] by showing that the elliptic potentials that lead to "rational" and "elliptic" Fuchsian equations can be integrated via one and the same "machinery". For both cases, we 1) embed the elliptic finite-gap theory into the context of the Galois theory and the Kovacic algorithm and describe 2) the corresponding differential Galois group, 3) the factorization of the operator (1), 4) the algebraic structure of the monodromy group, and 5) its matrix representations. From an algorithmic viewpoint, we present 6) a simple differentially-recursive recipe for the calculation of an object (the fundamental function $\boldsymbol{R}$) in terms of which all the listed above can be expressed.

The general finite-gap potentials also admit formulation in the language of the Picard–Vessiot theory [21], but as could be expected, for the elliptic solitons we have additional effects: they involve Fuchsian equations, monodromies, and algorithms. These objects were not touched upon in [20, 21], but those papers contain more thorough motivations and bibliography.

## §2. Finite-gap potentials

Among the finite-gap integration methods, there are several approaches that, in the final analysis, constitute parts of a general theory. For example, the (historically first) spectral viewpoint interprets the finite-gap potentials in problem (1) as having a finite number $g$ of forbidden gaps in their spectrum. The algebraic approach deals with a formally algebraic consideration of operators of the form $\partial_{xx} - u$ [4, 23]. The algebro-geometric (theta-functional) approach [5] employs the Riemann surfaces techniques intensively. All these aspects are summarized in the book [17]; it is known that the solution of problem (1) for any potential in this class is described by the famous theta-functional formulas [17, 5, 6]

$$u = -2\frac{d^2}{dx^2}\ln\Theta(\boldsymbol{U}x+\boldsymbol{D}) + \text{const}, \quad \Psi(x;\lambda) = \frac{\Theta\big(\boldsymbol{U}x+\boldsymbol{\mathfrak{U}}(\lambda)\big)}{\Theta(\boldsymbol{U}x+\boldsymbol{D})}\,\mathrm{e}^{\Omega(\lambda)x}\,, \tag{3}$$

where we shall need no specification for the $\Theta$-objects involved. However, these formulas need "to be made efficient", because the occurring parameters are rather complicated and are calculated, like the $\Theta$-function itself, in a quite a nontrivial way. Nevertheless, there is a particular, though very vied, class for which the expressions (3) reduce to the elliptic $\sigma$, $\zeta$, and $\wp$-functions, whose theory is developed fairly well. At the present time, the elliptic potentials constitute a theory of their own, namely, that of elliptic solitons (with numerous examples [16, 17, 11, 32]), which includes also the general complex version of the spectral theory for doubly periodic solutions [26, 27]. After the Lamé potentials, the most known example is the 2-gap potential of Darboux–Treibich–Verdier [17, 16, 38] $\phi(x) = 6\wp(x) + 2\wp(x-\omega)$, where $\omega$ is the half-period of the Weierstrass function. Below in the paper it will be shown that any finite-gap potential can be taken into some exactly solvable equation of the Fuchs class. Some of them fall into the pattern of a known algorithmic method, but most of them do not.

Any not theta-functional way of representing a solution for a finite-gap $\Psi$-function makes use of the Hermit equation [17]

$$\boldsymbol{R}_{xxx} - 4(u+\lambda)\boldsymbol{R}_x - 2u_x\boldsymbol{R} = 0, \tag{4}$$

and we shall see that this case is most naturally adapted to algorithmic methods. The corresponding formula looks like this [13]:

$$\Psi^{\pm}(x;\lambda) = \sqrt{\boldsymbol{R}(x;\lambda)}\exp\int^x \frac{\pm\mu\,dx}{\boldsymbol{R}(x;\lambda)}\,, \tag{5}$$

where $\mu$ is a constant arising after one-fold integration of equation (4) (see [4]):

$$\mu^2 = -\frac{1}{2}\boldsymbol{R}\boldsymbol{R}_{xx} + \frac{1}{4}\boldsymbol{R}_x^2 + (u+\lambda)\boldsymbol{R}^2\,, \tag{6}$$

this is an implicit form of the Wronskian of the solutions (5):

$$\Psi_x^+\Psi^- - \Psi^+\Psi_x^- = \sqrt{\boldsymbol{R}_x^2 - 2\boldsymbol{R}\boldsymbol{R}_{xx} + 4(u+\lambda)\boldsymbol{R}^2} = 2\mu(\lambda).$$

It is known that, for the potentials under consideration, the fundamental object $\boldsymbol{R}$ by which the solution (5) is constructed is a $\lambda$-polynomial of degree $g$ [17, 39, 20]:

$$\boldsymbol{R} = \lambda^g + R_1([u])\lambda^{g-1} + \cdots + R_g([u])\,, \tag{7}$$

and that the coefficients $R_k$ are differential polynomials in $u$, i.e.,

$$R_k = R_k(u, u_x, u_{xx}, \dots) =: R_k([u]).$$

They can be computed with the help of various recursive methods [4, 23]. Then, formula (6) turns into a dependence $\mu = \mu(\lambda)$ via a hyperelliptic curve of the form

$$\mu^2 = (\lambda - E_1)\dots(\lambda - E_{2g+1}), \tag{8}$$

and the number $g$, in the general position case, is the topological genus of that curve.

## §3. Elliptic potentials and Fuchsian equations

By definition, a differential equation $\Psi'' = Q(z)\Psi$, viewed as an equation on a manifold $\mathcal{M}$ with local/global coordinate $z$, belongs to the Fuchs class if the coefficient $Q(z)$ has only finitely many singularities on $\mathcal{M}$ each of which is a pole of order at most 2. The case of $\mathcal{M} = \bar{\mathbb{C}}$ delivers an example of a classical Fuchsian equation with rational coefficients.

Let $u = \phi(x)$ be an elliptic finite-gap potential, and let $u = Q_1(\wp) + Q_2(\wp)\wp'$ be its $\wp$-representation with some rational functions $Q_1$, $Q_2$. Here $\wp = \wp(x)$ is the standard Weierstrass function [1],

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e)(\wp - e')(\wp - e'')\,, \tag{9}$$

and $(g_2, g_3)$ or $(e, e', e'')$ are parameters. We change the variables by the rule

$$(x, \Psi) \mapsto (z, \psi): \quad z = \wp(x), \quad \psi = \sqrt{\wp'(x)}\Psi\,. \tag{10}$$

Using the identity

$$\Psi_{xx} = \wp'(x)^2\Psi_{zz} + \wp''(x)\Psi_z = (4z^3 - g_2 z - g_3)\Psi_{zz} + \frac{1}{2}(12z^2 - g_2)\Psi_z\,,$$

and making the above scale transformation $\Psi \mapsto \psi$, we see that equation (1) takes the form

$$\psi_{zz} = -\left\{\frac{3}{16}\frac{(4z^2 + g_2)^2 + 32g_3 z}{(4z^3 - g_2 z - g_3)^2} - \frac{Q_1(z) + wQ_2(z) + \lambda}{4z^3 - g_2 z - g_3}\right\}\psi\,, \tag{11}$$

where $w^2 := 4z^3 - g_2 z - g_3$. This equation can be viewed as an operator $\lambda$-pencil of the form of a generalized Sturm–Liouville problem

$$\psi_{zz} = \big\{p(z) + \lambda q(z)\big\}\psi \tag{12}$$

with coefficients $p$, $q$ lying on the torus $\big(z, w(z)\big)$, which is an algebraic elliptic Weierstrass curve. The theory of elliptic solitons provides many examples of elliptic finite-gap potentials (see, e.g., [16, 17, 11]). Taking one of them for the role of $\phi(x)$ and making the change (10), we get numerous examples of equations of the form (11), (12). Of course, changes of variables take solvable equations to solvable ones, but not every change results in a Fuchsian equation.

**Theorem 1.** *Equations* (11) *are Fuchsian on the torus* (9) *with possible reduction to the* $(z)$*-plane. In both cases, these equations admit integration in elliptic quadratures with a computable monodromy group.*

*Proof.* Formula (3) shows that any elliptic finite-gap potential has no residues; therefore, it is a "$\wp$-sum" over its poles:

$$u = \sum\nolimits_{\varrho} A_\varrho \wp(x - \varrho) + \mathrm{const} = Q_1(\wp) + Q_2(\wp)\wp'. \tag{13}$$

The potentials may be either even ($Q_2 = 0$), or not even ($Q_1, Q_2 \neq 0$). In the general position case, equation (11) is well defined on the 2-sheeted Riemann surface $\mathcal{R}$ of the Weierstrass equation (9), and the variable $x$ itself can be taken for the role of the (global) parameter on that surface. Since the passage $z \mapsto x$ is locally algebraic (the structure (2) is not lost), the fact that equation (11) is Fuchsian follows automatically from (13), because the potential in (13) has finitely many poles, all of the second order, on the quotient $\mathbb{C}/\{2\omega, 2\omega'\}$ of the $x$-plane by the periodicity lattice, i.e., on the parallelogram of periods. If needed, thus can be checked by analyzing the algebraic $(z, w)$-form (11) with the local parameters chosen appropriately.

If $Q_2 = 0$, the Fuchs property on the torus survives, but, by the independence of a $w$-sheet $\mathcal{R}$, equation (1), (13) is determined completely already on the half of the parallelogram of periods. It is well known that the function $\wp(x)$ maps this set conformally onto the entire plane $\mathbb{C}$, so that (1) turns into a Fuchsian equation on the extended $z$-plane $\bar{\mathbb{C}}$ via the substitution $\wp(x) = z$, or via its (nonessential) generalization of the form $\wp(x) = \frac{az+b}{cz+d}$.

More explicitly, in the case of an even potential we can write

$$Q_1(z) + Q_2(z)w = A_0\wp(x) + \sum\nolimits_\varrho A_\varrho\big\{\wp(x-\varrho) + \wp(x+\varrho)\big\} = \dots,$$

because the singularities arise in pairs $\pm\varrho$, and there is no loss of generality in assuming that among them we have $\varrho = \{\omega, \omega', \omega''\}$. Put $\wp_\varrho := \wp(\varrho)$ and introduce $\wp_\varrho'$, $\wp_\varrho''$ similarly. Applying the $\wp$-summation theorem, we pass to the variable $z = \wp(x)$:

$$\dots = A_0 z + \sum\nolimits_\varrho A_\varrho\bigg\{\frac{(\wp_\varrho')^2}{(z-\wp_\varrho)^2} + \frac{\wp_\varrho''}{z-\wp_\varrho} + 2\wp_\varrho\bigg\},$$

where we must put $\varrho = \{\omega, \omega', \omega''\}$ for $\wp_\varrho' = 0$. Plugging this in equation (11), we get poles of the second order, and as $z \to \infty$ we have

$$\psi_{zz} = \frac{1}{16}\big\{(4A_0 - 3)z^{-2} + \mathrm{O}(z^{-3})\big\}\psi.$$

Passing to the local coordinate $\xi = z^{-1}$, again we obtain a second order pole. Any potential $\phi(x)$ turning into an even one via a shift gives rise to an equation on the plane.

Once the potential (13) and the number $g$ are given, we may think of $\boldsymbol{R}([u];\lambda)$ as of a known rational function (7) of $(z, w, \lambda)$ (see Theorem 9 below). We suppress $\lambda$ in the notation and put $\boldsymbol{R}([u];\lambda) = \boldsymbol{R}_1(z) + \boldsymbol{R}_2(z)w$. Recalling (10), we see that formula (5) turns into the elliptic integral

$$\psi^\pm(z;\lambda) = \sqrt{(\boldsymbol{R}_1 + \boldsymbol{R}_2 w)w}\,\exp\int_a^z \frac{\mp\mu\boldsymbol{R}_2}{G}\,dz \cdot \exp\int_a^z \frac{\pm\mu\boldsymbol{R}_1}{G}\frac{dz}{w}\,, \tag{14}$$

where $G := \boldsymbol{R}_1^2(z) - (4z^3 - g_2 z - g_3)\boldsymbol{R}_2^2(z)$ and $\mu := \sqrt{(\lambda - E_1)\dots(\lambda - E_{2g+1})}$.

The branching character of the radical factor in (14) is known. The first integral in (14) is calculated elementarily, being an integral of a rational function, and standard actions reduce[1] both integrals to a linear combination of the canonical forms

$$P(z), \quad \int\frac{dz}{z-c}, \quad \int\frac{dz}{w}, \quad \int z\,\frac{dz}{w}, \quad \frac{1}{2}\int\frac{w+\wp'(\alpha)}{z-\wp(\alpha)}\,\frac{dz}{w}\,. \tag{15}$$

Discarding the rational part $P(z)$, we see that the monodromy $\mathfrak{G}_z$ is then determined by the integrals (15) over closed contours. If we choose, e.g., the lower limit $a = \{e,$ $e'$, $e''$, $\infty\}$, these integrals will become combinations of standard basis cycles on $\mathcal{R}$, and the generators $M$ of the group $\mathfrak{G}_z$ will be proportional to the exponentials of the periodicity modules of the integrals (15). In fact, it suffices to pass to the Weierstrass $\sigma$, $\zeta$-equivalents for (15) (see [1]) and take their independent "periods". They will be equal to the quantities

$$\{0\}, \quad \{2\pi\mathrm{i}\}, \quad 2\{\omega,\, \omega'\}, \quad 2\{\eta,\, \eta'\}, \quad 2\big\{\zeta(\alpha)\omega - \alpha\eta,\, \zeta(\alpha)\omega' - \alpha\eta'\big\} \tag{16}$$

[1]This process is automatized easily, with a by-product calculation of all constants.

for each type of the integrals in (15), respectively, where $\eta := \zeta(\omega)$ and $\eta' := \zeta(\omega')$. $\square$

*Remark* 1. The monodromy can be calculated much simpler, because it is a subgroup of the differential Galois group. This will be discussed in detail in Subsections 4.2 and 5.1.

## §4. Monodromy groups

**4.1. Structure of groups.** The above considerations touched upon "analytic" properties of groups. The assertions below show that, despite the fact that equations (11) can have any number of singularities, these groups have a relatively simple algebraic structure. Largely, this is explained by the existence of the explicit formula (14) for the $\psi$-function, i.e., by the finite-gap property. In the special case of the Lamé potentials, the algebraic structure of the monodromy group was established in [18, Theorem 3.3]. In what follows, we shall tacitly assume that we deal with the generic case, i.e., $\lambda \neq E_j \Leftrightarrow \mu \neq 0$.

**Theorem 2.** *For an arbitrary elliptic finite-gap potential, the monodromy* $\mathfrak{G}_z$ *of equation* (11) *is a nonfree group of rank* 3 *or* 5 *with the following systems of generators*[2]:

$$\begin{aligned} &Q_2 = 0: \quad \mathfrak{G}_z = \big\langle \mathfrak{a}^4, \mathfrak{b}^4, \mathfrak{c}^4, (\mathfrak{abc})^4 \big\rangle, \\ &Q_2 \neq 0: \quad \mathfrak{G}_z = \big\langle \mathfrak{a}^2, \mathfrak{b}^2, \mathfrak{c}^2, (ABA^{-1}B^{-1}\mathfrak{abc})^2 \big\rangle. \end{aligned} \tag{17}$$

*Proof.* We rewrite equation (11) indicating the singularities explicitly:

$$\begin{aligned} \frac{\psi_{zz}}{\psi} = &-\frac{3}{16}\left\{\frac{1}{(z-e)^2} + \frac{1}{(z-e')^2} + \frac{1}{(z-e'')^2}\right\} + \frac{1}{2w^2}(3z + 2A_0 z + 2\lambda) \\ &+ \frac{1}{2w^2}\cdot\sum_{\varrho\neq 0} A_\varrho \left\{\wp'_\varrho \frac{\wp'_\varrho + w}{(z-\wp_\varrho)^2} + \frac{\wp''_\varrho}{z-\wp_\varrho} + 2\wp_\varrho\right\}. \end{aligned} \tag{18}$$

By the finite-gap property, the coefficients $A_\varrho$ in (13) are integers of the form $n(n+1)$ [16, 17, 38]. Therefore, near a point $z = \wp_\varrho$ with $\varrho \neq \{\omega, \omega', \omega''\}$, instead of (18) we have

$$\psi_{zz} = \left\{\frac{n(n+1)}{(z-\wp_\varrho)^2} - (1\pm 1)\frac{\wp''_\varrho}{(\wp'_\varrho)^2}\frac{n(n+1)}{z-\wp_\varrho} + (\dots)_\pm\right\}\psi, \tag{19}$$

where the expansions $(\dots)_\pm$ are holomorphic, but depend on the number of the sheet. This yields the local structure of solutions

$$\begin{aligned} \psi_1 &= (z-\wp_\varrho)^{-n}\big\{1 + \alpha_1^\pm (z-\wp_\varrho) + \dots\big\}, \\ \psi_2 &= (z-\wp_\varrho)^{n+1}\big\{1 + \alpha_2^\pm (z-\wp_\varrho) + \dots\big\}. \end{aligned} \tag{20}$$

There will be no logarithms, because the substitution $z = \wp(x)$ is locally algebraic and the function $\Psi(x)$ (and, with it, $\psi(z)$) defined by formula (3) has no logarithmic singularities. Thus, actually, at the points $z = \wp_\varrho$ the solutions have no branching, i.e., their monodromies, generated by 1-fold bypassings of these points, are identical transformations. Composing the expansions for the remaining points $z = \{e, e', e'', \infty\}$, we shall have, for instance,

$$\psi_{zz} = \frac{1}{16}\left\{\frac{4A_\omega - 3}{(z-e)^2} + (\dots)_\pm\right\}\psi, \quad A_\omega = n(n+1).$$

Consequently, here the local solutions are given by convergent series

$$\begin{aligned} \psi_1 &= (z-e)^{\frac{1}{4}(3+2n)}\big\{1 + \alpha_1^\pm (z-e) + \dots\big\}, \\ \psi_2 &= (z-e)^{\frac{1}{4}(1-2n)}\big\{1 + \alpha_2^\pm (z-e) + \dots\big\} \end{aligned} \tag{21}$$

[2] "Genetic code", in the terminology of [9].

also without logarithms. As before, at infinity we have

$$\psi_{zz} = \frac{1}{16}\big\{(4A_0 - 3)z^{-2} + \mathrm{O}(z^{-3})_\pm\big\}\psi$$

with expansions similar to (21).

For the even potentials ($Q_2 = 0$), the local monodromy matrices $M$ are generated by 1-fold bypassings of the points $z = \{e, e', e'', \infty\}$. Therefore, the expansions (21) imply the following relations in the group $\mathfrak{G}_z$:

$$M_e^4 = M_{e'}^4 = M_{e''}^4 = M_\infty^4 = 1\,, \quad M_e M_{e'} M_{e''} M_\infty = 1. \tag{22}$$

In the case where $Q_2 \neq 0$, the loop bypassings of the points $\{e, e', e''\}$ must be 2-fold, so that the relations turn into involutions for each of the $M$-generators:

$$M_e^2 = M_{e'}^2 = M_{e''}^2 = M_\infty^2 = 1, \tag{23}$$

and their product must be equal to the commutant $[A, B] := ABA^{-1}B^{-1}$ of the fundamental generators on $\mathcal{R}$:

$$M_e M_{e'} M_{e''} M_\infty = [A, B].$$

So, in this case we get the group $\mathfrak{G}_z = \big\langle M_e^2, M_{e'}^2, M_{e''}^2, ([B, A]M_e M_{e'} M_{e''})^2\big\rangle$. Since the generators $A$ and $B$ are not local (they are not elliptic), they are not related to singularities of the Fuchsian equation under study. □

*Remark* 2. The rank of the monodromy does not grow when the number of the singularities $z = \wp_\varrho$ grows; therefore it is reasonable to ask whether such singularities, like false singularities, can be removed by a renormalization of the $\psi$-function, turning (11) into an equation with four singularities, e.g., into a Heun equation? A simple argument shows that this is impossible. Indeed, formulas (20) imply that for checking the removability of the singularity at $z = \wp_\varrho$ (denote $\boldsymbol{z} := z - \wp_\varrho$) it suffices to check the *Ansatz* $\psi = \boldsymbol{z}^{-n} \cdot \Xi$. Substituting it in (19), we obtain

$$\Xi'' - 2n\boldsymbol{z}^{-1}\Xi' - n(-n-1)\boldsymbol{z}^{-2}\Xi - \boldsymbol{z}^{-2}\big(n(n+1) + a\boldsymbol{z} + \dots\big)\Xi = \Xi'' - \frac{2n}{\boldsymbol{z}}\Xi' - \frac{1}{\boldsymbol{z}}(a + \dots)\Xi = 0.$$

Consequently, for the absence of singularities in the coefficients it is necessary that, at least, $n = 0$, i.e., the singularity $z = \wp_\varrho$ is absent at all.

The normal form of equations (11) was chosen because it is unique. However, it implies the many-valued factor $\sqrt{w}$ in the solution $\psi$. The branching character of this factor (viewed as an algebraic function) does not depend on $z$; therefore, we can discard it, renormalizing the $\psi$-function and sacrificing the normal form. Then, the monodromy groups take even simpler (canonical) structure.

**Theorem 3.** *Renormalize the $\psi$-function by the rule $\psi \mapsto \Xi$:*

$$\psi = \sqrt[4]{(z - e)(z - e')(z - e'')} \cdot \Xi\,. \tag{24}$$

*Then for an arbitrary elliptic finite-gap potential, the monodromy group of $\Xi$-solutions is either the 2-dimensional crystallographic group* $\mathbf{p2}$ *of rang 3, or its selfadjoint subgroup of index 3, the commutative group of the 1-dimensional torus.*

*Proof.* After the renormalization (24), formulas (21) turn into the expansion

$$\Xi_1 = (z - e)^{\frac{n+1}{2}}\{1 + \dots\}, \quad \Xi_2 = (z - e)^{-\frac{n}{2}}\{1 + \dots\}. \tag{25}$$

Therefore, for $Q_2 = 0$, relations (22) are taken to (23), and we get a group whose 2-dimensional crystallographic origin is well known. This is the group

$$\mathbf{p2} = \big\langle M_e^2, M_{e'}^2, M_{e''}^2, (M_e M_{e'} M_{e''})^2\big\rangle, \tag{26}$$

see [9, p. 41, (4.502)]. Obviously, no other relations arise. In the case where $Q_2 \neq 0$, we deal with two copies of the $z$-plane $\bar{\mathbb{C}}$ with due rules of the passage between sheets. Consequently, relations (23) disappear, turning into the identical transformations $M_e = M_{e'} = M_{e''} = M_\infty = 1$, and the two remaining generators become two commuting ones; thus, they generate the torus group $\mathfrak{T} = \langle [A, B] \rangle$.

For simplicity, we redenote the $M$-transformations by $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$. To establish a relationship between **p2** and $\mathfrak{T}$, we construct the left shift of the generators of **p2** by one of them:

$$U := \mathfrak{ab}, \quad S := \mathfrak{ac}, \quad 1 = \mathfrak{aa}$$

and form the group $\langle U, S \rangle =: H$. Since $\mathfrak{bc} = \mathfrak{ba} \cdot \mathfrak{ac} = U^{-1}S$, we see that any product of an even number of elements of **p2** can be expressed in terms of elements of $H$. For example,

$$(\mathfrak{ab}, \mathfrak{ac}, \mathfrak{bc}) = (U, S, U^{-1}S) \text{ and } (\mathfrak{ba}, \mathfrak{ca}, \mathfrak{cb}) = (U^{-1}, S^{-1}, S^{-1}U).$$

Consequently, for any $\alpha \in \mathbf{p2}$ the expression $\alpha H \alpha^{-1}$ consists of products of an even number of elements $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, so that it belongs to $H$. Two situations are possible: $H \subset \mathbf{p2}$ and $H = \mathbf{p2}$, depending on the presence/absence of relations supplementary to the involutions $\mathfrak{a}^2 = \mathfrak{b}^2 = \mathfrak{c}^2 = 1$, and the group $H$ itself may be commutative or not[3]. In our case, all defining relations have even length; therefore, each word in **p2** has an invariable parity of its length, whence $H \neq \mathbf{p2}$. The group is split into $H$ and the odd length elements $\mathfrak{a}H$, $\mathfrak{b}H$, $\mathfrak{c}H$. In its turn, $\mathfrak{b}H = \mathfrak{a}^2\mathfrak{b}H = \mathfrak{a}UH = \mathfrak{a}H$, and the same for $\mathfrak{c}H$. Thus, the group **p2** consists of $H$ and the class $\mathfrak{a}H$, i.e., $|\mathbf{p2} : H| = 2$, which proves selfadjointness. The commutativity of $H$ will follow if we exclude the last defining relation $(\mathfrak{abc})^2 = 1$. Indeed, $(\mathfrak{abc})^2 = \mathfrak{ab} \cdot \mathfrak{ca} \cdot \mathfrak{bc} = U \cdot S^{-1} \cdot U^{-1}S = 1$, i.e., $US = SU \Longrightarrow H = \langle [U, S] \rangle = \mathfrak{T}$. This gives also a way to construct generators: $(A, B) = (\mathfrak{ab}, \mathfrak{ca})$. ☐

*Remark* 3. The groups listed above coincide formally with Fuchsian type groups with finite topological genus $g = 0, 1$. Nevertheless, neither factorization by the center (matrices of the form $\mathrm{Diag}(\alpha, \alpha)$), nor representations by groups of linear-fractional transformations, makes any sense, because inversion of the ratio $\psi_1(z)/\psi_2(z)$ of "finite-gap" solutions is never a one-valued function.

**4.2. Representations of monodromies.** The expansions (25) show that if $Q_2 = 0$, then we can always find a pair of solutions for which one of monodromy involutions has the simplest (canonical) form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Under the second of these two normalizations, monodromy was expressed in terms of elliptic integrals in [38] in the particular case where (11) is a Heun equation, i.e., $u$ is one of the Darboux–Treibich–Verdier potentials, and the collection of points $\{e, e', e''\}$ is equivalent to a real collection.

On the other hand, if we know that all groups have a relatively simple structure, then it is natural to expect that the general form of "finite-gap" monodromies can be found. This can be done easily as soon as a formula for the solution is known.

**Theorem 4.** *For any even elliptic finite-gap potential $u = Q_1(z)$, the monodromy group for the equation* (*the pair of solutions $\Xi_\pm(z) = \Psi_\pm(x)$*)

$$(4z^3 - g_2 z - g_3)\Xi'' + \frac{1}{2}(12z^2 - g_2)\Xi' = (Q_1(z) + \lambda)\Xi \quad (\lambda \neq E_k) \tag{27}$$

*is isomorphic to the representation of the group* **p2** *on the antidiagonal matrices*

$$M_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_{e'} = \begin{pmatrix} 0 & p \\ p^{-1} & 0 \end{pmatrix}, \quad M_{e''} = \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix} \tag{28}$$

[3] It is not hard to present examples where all these possibilities are realized.

*(permutational involutions), where*

$$p := \exp \int_{e}^{e'} \frac{2\mu}{\boldsymbol{R}(z)} \frac{dz}{w}, \quad q := \exp \int_{e}^{e''} \frac{2\mu}{\boldsymbol{R}(z)} \frac{dz}{w} \tag{29}$$

*for the corresponding basis of solutions* $\{\Xi_+, \Xi_-\}$.

*Proof.* We write the solutions of equation (27) for $\mu \neq 0$

$$\Xi_{\pm} = \exp \int_{z_0}^{z} \frac{w\boldsymbol{R}_z \pm 2\mu}{2\boldsymbol{R}} \frac{dz}{w} \tag{30}$$

and employ the fact that, for the even potentials, $\boldsymbol{R}$ is a function of $z$ only [39]. In Subsection 5.2 we shall give yet another proof of this property, together with a direct method for calculating $\boldsymbol{R}(z)$. Then, formula (30) can be rewritten as

$$\Xi_{\pm}(z) = \exp \int_{z_0}^{z} \big\{ w\boldsymbol{r}(z) \pm \mu\boldsymbol{s}(z) \big\} \frac{dz}{w} \tag{31}$$

with obvious expressions for the rational functions $\boldsymbol{r}(z)$ and $\boldsymbol{s}(z)$. We make a one-loop tour around, e.g., the point $e$ only. The form of the loop is immaterial because, by Theorem 2, the solutions $\{e, e', e'', \infty\}$ can ramify only at the points $\Xi_{\pm}(z)$. We write

$$\begin{aligned}
\Xi_{\pm}(z) \mapsto \exp &\left\{ \int_{z_0}^{z} (w\boldsymbol{r} \pm \mu\boldsymbol{s}) \frac{dz}{w} + \int_{z}^{e} (w\boldsymbol{r} \pm \mu\boldsymbol{s}) \frac{dz}{w} + \int_{e}^{z} (-w\boldsymbol{r} \pm \mu\boldsymbol{s}) \frac{dz}{-w} \right\} \\
&= \exp \left\{ \int_{z_0}^{e} (w\boldsymbol{r} \pm \mu\boldsymbol{s}) \frac{dz}{w} + \int_{e}^{z} (w\boldsymbol{r} \mp \mu\boldsymbol{s}) \frac{dz}{w} \right\} \\
&= \Xi_{\pm}(e) \cdot \exp \int_{e}^{z} (w\boldsymbol{r} \mp \mu\boldsymbol{s}) \frac{dz}{w} \\
&= \Xi_{\pm}(e) \cdot \exp \int_{e}^{z_0} (w\boldsymbol{r} \mp \mu\boldsymbol{s}) \frac{dz}{w} \cdot \exp \int_{z_0}^{z} (w\boldsymbol{r} \mp \mu\boldsymbol{s}) \frac{dz}{w} = \frac{\Xi_{\pm}(e)}{\Xi_{\mp}(e)} \cdot \Xi_{\mp}(z),
\end{aligned}$$

which is as yet formal, because, generally speaking, the integrals with limit $e$ diverge, and, by (25) the solutions may tend to zero or infinity, and the ratio $\Xi_{\pm}(e)/\Xi_{\mp}(e)$ may fail to be defined. However, monodromy is always a nonsingular transformation, so that $\Xi_{\pm}(e)/\Xi_{\mp}(e) \neq \{0, \infty\}$. Thus, using (30), we see that the object

$$\frac{\Xi_{\pm}(\boldsymbol{e})}{\Xi_{\mp}(\boldsymbol{e})} = \exp \int_{z_0}^{\boldsymbol{e}} \frac{\pm 2\mu}{\boldsymbol{R}(z)} \frac{dz}{w} \tag{32}$$

is a nonzero finite quantity[4] at any of the points $\boldsymbol{e} = \{e, e', e'', \infty\}$ ($z_0$ is viewed as a general position point). Consequently,

$$\begin{pmatrix} \Xi_+ \\ \Xi_- \end{pmatrix} \mapsto \begin{pmatrix} 0 & p \\ p^{-1} & 0 \end{pmatrix} \begin{pmatrix} \Xi_+ \\ \Xi_- \end{pmatrix}, \quad p := \exp \int_{z_0}^{e} \frac{2\mu}{\boldsymbol{R}(z)} \frac{dz}{w}.$$

This proves that each generator is a permutational involution. The last algebraic relation in (26) for **p2** can be checked directly. Putting $z_0 = e$, we obtain the generators (28). $\square$

The above proof opens way to find monodromy also in other bases, but in any case the procedure reduces to calculating the exponentials of the differences of elliptic integrals as in (32) at the points $\{e, e', e''\}$ (this was mentioned in Theorem 1), which are proportional to the quantities (16). Sometimes, the answers are expressed in terms of the solutions $\Xi_{\pm}(z)$ themselves. The way to obtain these solutions with the help of Jacobi–Weierstrass

[4] In particular, this implies the fact that $\boldsymbol{R}(z)$ has no zeros at the points $z = \{e, e', e''\}$ (cf. the special case of Heun's equation treated in [38, Corollary 3]). Otherwise, the integral in (32) would be meromorphic at the points $\boldsymbol{e} = \{e, e', e''\}$, which contradicts the above-mentioned property of the ratio $\Xi_{\pm}(e)/\Xi_{\mp}(e)$.

functions is well known in the theory of elliptic solutions (see [12, 17, 16]); these are the Hermite–Halphen $\theta$- and $\sigma$-*Ansätze.* Therefore, we illustrate the said above, omitting the calculations.

### 4.3. Comments and examples.

**Example 1.** The Lamé potential $u = 2\wp(x)$ for $\lambda \neq \{e, e', e''\}$. Here we deduce that, in the basis

$$\Xi_{\pm}(z) = \exp \frac{1}{2} \int_e^z \frac{w \pm \mu}{z - \lambda} \frac{dz}{w} \tag{33}$$

monodromy has the **p2** -form (28), where

$$\begin{aligned} p &:= \exp 2\big\{\zeta(\alpha)(\omega - \omega') - \alpha(\eta - \eta')\big\}, \\ q &:= \exp 2\big\{\zeta(\alpha)(\omega + 2\omega') - \alpha(\eta + 2\eta')\big\} \end{aligned} \tag{34}$$

and $\alpha = \wp^{-1}(\lambda)$. If $\mu = 0$, e.g., $\lambda = e$, we choose the basis of solutions

$$\Xi_-(z) = \sqrt{z - e}, \quad \Xi_+(z) = \sqrt{z - e} \int_{z_0}^z \frac{1}{z - e} \frac{dz}{w}.$$

By analytic continuation, we find that, like in the preceding case of $\mu \neq 0$, the entries of the monodromy matrices are expressed in terms of the values of solutions at the points $\{e, e', e''\}$:

$$M_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_{e'} = \begin{pmatrix} -1 & p \\ 0 & 1 \end{pmatrix}, \quad M_{e''} = \begin{pmatrix} -1 & q \\ 0 & 1 \end{pmatrix},$$

where

$$p := 2\frac{\Xi_+(e')}{\sqrt{e' - e}}, \quad q := 2\frac{\Xi_+(e'')}{\sqrt{e'' - e}}.$$

Now we turn to a transcendental parameter $\mathfrak{u}$ of elliptic functions and, therefore, put $z_0 = \wp(\mathfrak{u}_0)$. To simplify expressions, for the role of $\mathfrak{u}_0$ we choose the solution of the transcendental equation $\zeta(\omega - \mathfrak{u}_0) = e\mathfrak{u}_0$. Then calculations yield the expressions

$$\begin{aligned} \Xi_+(e') &= \frac{\sigma(\omega'')}{\sigma(\omega)\sigma(\omega')} \frac{\eta - \eta' - e\omega'}{(e' - e)(e'' - e)} \,\mathrm{e}^{-\eta\omega'}, \\ \Xi_+(e'') &= \frac{\sigma(\omega')}{\sigma(\omega)\sigma(\omega'')} \frac{\eta'' - \eta + e\omega''}{(e' - e)(e'' - e)} \,\mathrm{e}^{-\eta\omega''} \end{aligned} \tag{35}$$

(they will look even more compact if we translate them in the language of the Jacobi $\vartheta$-coordinates). This group is no longer of permutation type. For other even potentials, for $\lambda = E_j$ the monodromies can be computed similarly.

**Example 2.** The Lamé potential $u = 6\wp(x)$ with $\lambda$ distinct from the branching points of the curve

$$\mu^2 = (\lambda^2 - 3g_2)(\lambda + 3e)(\lambda + 3e')(\lambda + 3e'').$$

If we use the known basis of solutions

$$\Psi_{\pm}(x) = \frac{d}{dx}\frac{\sigma(x \mp \alpha)}{\sigma(x)} \,\mathrm{e}^{\pm(\zeta(\alpha)+k)x} = \left\{ \frac{1}{2} \frac{\wp'(x) \pm \wp'(\alpha)}{\wp(x) - \wp(\alpha)} \pm k \right\} \frac{\sigma(x \mp \alpha)}{\sigma(x)} \,\mathrm{e}^{\pm(\zeta(\alpha)+k)x} = \dots,$$

it suffices to rewrite it in the $z$-representation

$$\begin{aligned} \dots = \Xi_{\pm}(z) &= \left\{ \frac{1}{2} \frac{w \pm \wp'(\alpha)}{z - \wp(\alpha)} \pm k \right\} \exp \int^z \left\{ \frac{1}{2} \frac{w \pm \wp'(\alpha)}{z - \wp(\alpha)} \pm k \right\} \frac{dz}{w} \\ &= \left| \aleph := \frac{1}{2} \frac{w \pm \wp'(\alpha)}{z - \wp(\alpha)} \pm k \right| = \aleph \exp \int^z \aleph \,\frac{dz}{w} = \exp \int^z \left\{ \frac{w}{\aleph}(\aleph_z + w_z \aleph_w) + \aleph \right\} \frac{dz}{w} \end{aligned}$$

and, reshaping this to the form (31), to act as before. The algebraic relations among the quantities $\mu$, $\lambda$, and $k$, as well as their expressions in terms of $\alpha$ are also well known [16, 17]. Also, we can use formulas (28)–(29) directly, because for this and many other classical potentials the $\boldsymbol{R}$-function is either known, or easily calculated (see [17, 39]):

$$\boldsymbol{R} = 9z^2 - 3\lambda z + \lambda^2 - \frac{9}{4}g_2 = 9(z - \varkappa_+)(z - \varkappa_-)\,,$$
$$\varkappa_\pm = \frac{1}{6}\big(\lambda \pm \mathrm{i}\sqrt{3\lambda^2 - 9g_2}\big).$$

Then the integral (32) takes the form of the sum of two standard logarithmic elliptic integrals

$$\int \frac{2\mu}{\boldsymbol{R}(z)}\,\frac{dz}{w} = \frac{2}{9}\frac{\mu}{\varkappa_+ - \varkappa_-}\int \left\{\frac{1}{z - \varkappa_+} - \frac{1}{z - \varkappa_-}\right\}\frac{dz}{w}.$$

Since all components involved in these elliptic integrals are written in terms of $\wp$, $\wp'$, $\zeta$, and the logarithms of $\sigma$-functions, the monodromy coefficients will admit expressions in terms of differences of such representations taken at the points $\omega$, $\omega'$, $\omega''$.

*Remark* 4 (**Example 3**). In some cases, equation (11) under study reduces to other known equations, so that its group can be found from other considerations. The nearest (nonunique) example is the case where $g_2 = 0$. Next, we consider an arbitrary (not necessarily finite-gap) Lamé potential $Q_1(z) = Az$ and pass to the variable $w$. After the renormalization $\psi \mapsto \boldsymbol{\psi}$ of the form $\boldsymbol{\psi} = z\,{}^{-2}\!\!\sqrt{w}\cdot\psi$ of the $\psi$-function, we deduce the following simple equation on the elliptic equianharmonic curve:

$$\boldsymbol{\psi}_{ww} = \frac{1}{9}\frac{4\lambda z^2 + (A-2)w^2 + (A+6)g_3}{(w^2+g_3)^2}\boldsymbol{\psi}\,. \tag{36}$$

This suggests putting $\lambda = 0$, because in this case it is seen readily that the equation can be solved automatically in ${}_2F_1$-functions and even in the Legendre functions $P_\nu^\mu$, $Q_\nu^\mu$ [12]. Calculations yield

$$\boldsymbol{\psi}_{1,2} = \sqrt{w^2 + g_3}\left\{P_\nu^\mu\left(\tfrac{\mathrm{i}w}{\sqrt{g_3}}\right),\ Q_\nu^\mu\left(\tfrac{\mathrm{i}w}{\sqrt{g_3}}\right)\right\}, \tag{37}$$

where

$$\nu = \frac{1}{6}\sqrt{4A+1} - \frac{1}{2}, \quad \mu = \frac{1}{3}.$$

Here the parameter $A$ is arbitrary; it can be viewed as a new spectral parameter in the corresponding spectral problems of the form

$$\Psi'' = Au\Psi$$

instead of the "lost" $\lambda$. The monodromies of the functions (37) can be calculated by the receipts known from the theory of the hypergeometric equation, and they are more or less standard. It should be noted that, though the corresponding equation $\Psi'' = A\wp(x; 0, g_3)\Psi$ is solvable in Legendre functions, the finite-gap case where $A = n(n+1)$ stands somewhat apart: integrability in special functions (37) and integrability in quadratures (14) are different in the main, see [20, 21]. As another example of ${}_2F_1$-reducibility, we mention the lemniscate case, where $g_3 = 0$. Then the equation $\Psi'' = A\wp(x; g_2, 0)\Psi$ is taken to a hypergeometric one by another substitution [28]. It can be shown that this equation can also be solved in Legendre functions. In the finite-gap case where $A = n(n+1)$, equation (36) (and its extension with $g_2 \neq 0$) can be solved finitely, in the form written above, for any $\lambda$.

The above examples have an interesting consequence.

**Proposition 5.** *The Legendre functions $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ with*

$$\left\{\nu=\frac{1}{3}(n-1),\ \mu=\frac{1}{3}\right\},\quad \left\{\nu=\frac{1}{4}(2n-1),\ \mu=\frac{1}{4}\right\},\quad n\in\mathbb{Z},$$

*and their ${}_2F_1$-equivalents admit representations by indefinite elliptic integrals corresponding to the irrationalities*

$$\sqrt{4z^3-1}\ \text{ and }\ \sqrt{4z^3-z}\,.$$

*Proof.* The first case pertains to the finite-gap series (37), and the second comes from the equation $\Psi''=n(n+1)\wp(x;1,0)\Psi$ and the substitution $z=\wp^2(x;1,0)$, which is equivalent to the substitution found in [28]. □

These representations, apparently, are not universal, but they are of recursive nature, being based on the calculation of the function $\boldsymbol{R}$; this calculation will be given below in Subsection 5.2. The fact that the hypergeometric and Legendre functions can be expressed (and defined) in terms of contour integrals is standard [12], so that the property indicated above is far from being evident; to the best of our knowledge, it is not commonly known in the theory under consideration.

*Remark* 5. The renormalization (24) leads easily to the representation of the monodromy group (17) for equations in the normal form, i.e., equations (11). The renormalization itself can be varied, but if it changes the nature of ramification, then the group is no longer canonical. Otherwise, the structure of the group is kept. Curiously, the latter is possible even if renormalization involves not only rational functions. This is the case, e.g., in the passage to $\psi$-functions used in [38, 39]. Therefore, the monodromy representation there will be the same canonical **p2** as described above, and the restrictions imposed in [38] on the potentials, as well as the requirement that the numbers $n$, $n'$, $n''$ be real and positive, may be lifted.

Monodromy depends heavily on the choice of variables, and changes of variables (finite-sheeted coverings) lead almost always to modification of monodromy, and even of its rank. For example, this rank is equal to three for the group $\mathfrak{G}_z$ of equation (27), while for the $\mathfrak{G}_x$-monodromies of equations (1) it is equal to two. Clearly, in one direction, the generators can be expressed in terms of the other generators, e.g., $\mathfrak{G}_x$ in terms of $\mathfrak{G}_z$, but not *vice versa*. For the monodromy $\mathfrak{G}_x$, examples with calculations can be found in [40]. After bulky formulas, the results are expressed in terms of (not calculated) hyperelliptic integrals (see [40, formulas (3.36), (3.42)]), and the author of [40] says that "the calculation of the integral would be difficult". It is hard to agree with this because, by (14), the hyperelliptic form of integrals even does not arise in the theory of elliptic solitons, and already in [38] monodromy was written in terms of elliptic integrals. The hyperelliptic integrals will lead to elliptic ones again (reduction problem), and the proof of Theorem 1) shows that the techniques of calculating the latter integrals does not depend on what equation, (1), (11), or (27), is chosen for calculating the group. The final answers are brought to "elliptic $\sigma,\zeta,\eta,\omega$-formulas" of type (15), (34), (35) with the help of standard actions [1, §17]. Also note that the $\mathfrak{G}_x$-monodromy has an important physical interpretation of dispersion relations, and in this context it was calculated for Lamé potentials of equation (1) in the paper [33], where extensive references can also be found.

*Remark* 6 (**Example 4**). The infinite group $\mathfrak{G}_z$ may reduce to a finite one for certain values of the parameter $\lambda$. This happens if and only if both solutions are algebraic functions. This case was studied intensely and found many applications, see, e.g., [12, §23.41, 23.7] and the papers [32, 18] together with references therein. It is of interest to note that algebraic reductions are possible not only for $\mu = 0$, but also at some generic points $\lambda \neq E_j$. As a simplest example we mention the “1-gap version” of equation (36). There, in addition to $A = 2$, we put $\lambda = 0$ and $(g_2, g_3) = (0, 1)$; this will simplify formulas without any loss of generality. Then we get the equation and its solutions

$$\boldsymbol{\psi}_{ww} = \frac{8}{9}\frac{1}{(w^2+1)^2}\boldsymbol{\psi}, \quad \boldsymbol{\psi}_\pm = \sqrt[3]{(w^2+1)(w\pm \mathrm{i})},$$

which can be converted easily into algebraic solutions of the corresponding equations (11) and (1):

$$\psi_{zz} = \frac{5z^4 - 8z}{(4z^3-1)^2}\psi, \quad \psi_\pm = \sqrt[4]{4z^3-1}\sqrt[3]{\sqrt{4z^3-1}\pm \mathrm{i}}\,,$$

$$\Psi'' = 2\wp(x)\Psi, \quad \Psi_\pm = \sqrt[3]{\wp'(x)\pm \mathrm{i}}.$$

By the way, this is equivalent to the fact that the elliptic logarithmic integrals (33) are calculated in elementary functions if the parameters are as above, i.e., $w^2 = 4z^3 - 1$:

$$\int^z \frac{w\pm\mu}{z-\lambda}\frac{dz}{w} \quad \dashrightarrow \quad \int^z \frac{w\pm \mathrm{i}}{z}\frac{dz}{w} = \frac{2}{3}\ln(w\mp \mathrm{i})\,.$$

We have presented this curious example because, practically, there is no mention of it in the literature on the Lamé potentials. This example is far from being unique, and we note that algebraic solutions can occur not only for the “rational” equations (11), but also in the general case where $Q_2 \neq 0$. This is clear, because, e.g., the two variables (substitutions) $z = \wp(x)$ and $\boldsymbol{z} = \wp(x+\delta)$ are related to each other algebraically, $F(z, \boldsymbol{z}) = 0$.

## §5. Relationship with algorithms

Classical algorithms for integration of linear differential equations (among them, Singer’s method [35] and Kovacic’s method [31] should be mentioned first) apply to equations with rational coefficients. As far as we know, the methods for equations with nontrivial ($g > 0$) algebraic coefficients are far from efficient realization, especially if the equations involve parameters. Therefore, if an algorithm is applied to an equation of the form (11) (quite a wide class, having a great number of parameters) and integrates this equation for an arbitrary value of the parameter[5] $\lambda$, then this algorithm can be viewed as another representation of finite-gap integrability. Thus, the preceding sections show in fact that there is no difference between the algebraic and rational coefficients.

In other words, the $\Theta$-function formulas (3) of Its and Matveev provide solutions also in the algorithmic context, though the properties of being “quadrature” and algorithmic are fairly hidden for the $\Theta$-methods. These properties become more transparent if we supplement formulas (4)–(7) with the standard attributes of the differential Galois theory.

Recall that the Picard–Vessiot extension (see [30, 8] and [35, p. 12]) is constructed by joining the solutions of an equation to the differential field $F$ over which the equation is given. In our case, for equation (11), the field $F$ is $\mathbb{C}(z)$ or $\mathbb{C}(z, w)$. This extension is unnecessarily large (the decomposition field). As a nearest special type of extensions we mention the Liouville ones, see [30], [2, 53] and [35, p. 33] They serve as a subject of the classical algorithms [36, 31] and are constructed by recursive adjoining of an algebraic element, the integral, and the exponential of the integral. More detailed expositions can be found in the references cited.

[5] This is a distinguishing characteristic of the finite-gap potentials [20]: equations (11) are integrated in quadratures for all $\lambda$, and the dependence $\Psi(\lambda)$ is analytic and known.

**5.1. Factorization and the Galois group.** The integrability of a linear equation is related to its factorization into products of first order operators, see, e.g., [2]. In particular, a formal factorization of the operator (1) was used in [3], which led there to finite-gap potentials as well as those related to the 4th and 5th Painlevé transcendent $\mathrm{P}_4, \mathrm{P}_5$. From the differential Galois theory viewpoint, neither the first, nor the second case is exceptional, because any linear equation is factorized in the Picard–Vessiot extension[6]. Below, we specify this property and the Galois group for the (elliptic) finite-gap class, together with its transformation under the substitution $z = \wp(x)$.

**Proposition 6.** *Factorization of the Fuchsian equation* (11) *and the Sturm–Liouville operator in the finite-gap class looks like this*:

$$\partial_{xx} - (u+\lambda) = \left(\partial_x + \frac{1}{2}\frac{\boldsymbol{R}_x}{\boldsymbol{R}} \pm \frac{\mu}{\boldsymbol{R}}\right)\left(\partial_x - \frac{1}{2}\frac{\boldsymbol{R}_x}{\boldsymbol{R}} \mp \frac{\mu}{\boldsymbol{R}}\right).$$

*In particular, for the elliptic finite-gap potentials, equations* (1) *are factorized in elliptic functions, and equations* (11), (18) *are factorized over the field* $\mathbb{C}(z,w)$:

$$\begin{aligned}&\partial_{zz} + \frac{3}{16}\frac{(4z^2+g_2)^2+32g_3z}{(4z^3-g_2z-g_3)^2} - \frac{Q_1(z)+wQ_2(z)+\lambda}{4z^3-g_2z-g_3}\\ &\quad= \left(\partial_z + \frac{1}{4}\ln_z w^2 + \frac{1}{2}\frac{\boldsymbol{R}_z}{\boldsymbol{R}} \pm \frac{\mu}{w\boldsymbol{R}}\right)\left(\partial_z - \frac{1}{4}\ln_z w^2 - \frac{1}{2}\frac{\boldsymbol{R}_z}{\boldsymbol{R}} \mp \frac{\mu}{w\boldsymbol{R}}\right).\end{aligned} \tag{38}$$

*Proof.* The claims follow from the known fact [7, §17] that factorization is realized with the help of the logarithmic derivative of any particular solution, i.e., $\partial_{xx} - U = (\partial_x + \ln_x\varphi)(\partial_x - \ln_x\varphi)$ if $\varphi_{xx} = U\varphi$. For the role of $\varphi$ we choose the function (5). If $u(x)$ is an elliptic function, then $\boldsymbol{R}$ (see (7)) will also be an elliptic function. Formula (38) is less obvious, because the substitution (10) involves the factor $\sqrt{w}$ that does not belong to $\mathbb{C}(z,w)$. However, any radical belongs to the Liouville extension; therefore, bringing $\sqrt{w}$ under the sign $\exp\int$ and simplifying, we get (38). If $\mu \neq 0$, then factorization is always realized over $\mathbb{C}(z,w)$ and never over $\mathbb{C}(z)$, independently of the parity of the potential. ☐

Now we present a general characteristic of the differential Galois group for finite-gap Fuchsian equations.

**Theorem 7.** *In the general position case* ($\lambda \neq E_j$), *independently of the parity of the potentials* (13), *the Picard–Vessiot extension for the "finite-gap" equations* (11)⇔(18) *is a Liouville extension of the field* $\mathbb{C}(z,w)$, *and the group of differential automorphisms of equation* (11) (*Galois group*) *is connected and is similar to the group of matrices* $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$, *where* $\alpha \in \mathbb{C}\setminus\{0,\infty\}$.

*Proof.* The fact that the extension in question is Liouville follows from Proposition 6 and formula (38). The structure of the differential Galois group, by its definition, is found from the condition that all differentially relations among solutions and their derivatives are preserved under the linear basis transformations $(\psi^+,\psi^-) \mapsto (\psi^+,\psi^-)\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$. In our case, we have three (algebraically) independent relations

$$\psi^+\psi^- = w\boldsymbol{R}, \quad \psi_z^{\pm} = \frac{(w\boldsymbol{R})_z \pm 2\mu}{2w\boldsymbol{R}}\psi^{\pm}, \tag{39}$$

which, like equation (11) itself, are viewed as given over $\mathbb{C}(z,w)$. Checking the invariance of the second relations and recalling that $\mu \neq 0$, we see that $\beta = \gamma = 0$. This cuts off the

[6] In the classical paper [24], the 1st Painlevé transcendent $\mathrm{P}_1$ was mentioned.

disconnected component of the group, which is allowed formally by the first relation[7]. In its turn, the first relation yields $\delta = \alpha^{-1}$. The constant $\alpha$ is arbitrary and cannot be an algebraic number in the general position case; otherwise all the solutions would be algebraic. □

In the case of even potentials ($Q_2 = 0$), the differential field over which the equation may be given admits the restriction $\mathbb{C}(z,w) \to \mathbb{C}(z)$, so that the above Galois group may have extensions, remaining, of course, solvable. This is always the case indeed.

**Theorem 8.** *Let $u = Q_1(z)$ (even finite-gap potentials), and let equation* (11) *be defined over $\mathbb{C}(z)$. Then in the generic case ($\lambda \neq E_j$), the Picard–Vessiot extension is a Liouville extension, and the Galois group is isomorphic to the infinite (disconnected) dihedral group*

$$D_\infty = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & -\beta \\ \beta^{-1} & 0 \end{pmatrix} \right\}, \quad \alpha, \beta \in \mathbb{C} \setminus \{0, \infty\}.$$

*Proof.* As before, the Liouville property is obvious, but relations (39) are defined not over $\mathbb{C}(z)$. We write an equivalent of the second relation in (39), which is not reducible over $\mathbb{C}(z)$. Since $\boldsymbol{R} \in \mathbb{C}(z)$ for even potentials, we have

$$\psi_z^{\pm} = \frac{(w\boldsymbol{R})_z \pm 2\mu}{2w\boldsymbol{R}} \psi^{\pm} \quad \Longrightarrow \quad \left\{ 2\frac{\psi_z^{\pm}}{\psi^{\pm}} - \frac{w_z}{w} - \frac{\boldsymbol{R}_z}{\boldsymbol{R}} \right\}^2 = \frac{4\mu^2}{w^2 \boldsymbol{R}^2}. \tag{40}$$

Clearly, the two sides of the latter identity are already defined over $\mathbb{C}(z)$; checking invariance as before, we see at once a transformation that interchanges the $\psi$-components, $(\psi^+, \psi^-) \mapsto (\beta\psi^-, \gamma\psi^+)$. Recalling that $\mu \neq 0$, we conclude that then the possible transformations are the matrices of the form

$$\begin{pmatrix} \pm\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm\beta \\ \beta^{-1} & 0 \end{pmatrix}$$

because relations (40) are of quadratic nature. The same relations together with the quadratic version of the first equation in (39) imply the relation (Wronsckian)

$$\psi_z^+ \psi^- - \psi^+ \psi_z^- = 2\mu,$$

which leaves only the upper sign in $\pm\alpha$ and the lower sign in $\pm\beta$. As a result, we arrive at the group mentioned in the theorem. □

*Remark* 7. It should be noted that the canonical solutions (5) correspond to the forms of the monodromy generators (28) that are contained completely in the disconnected component of the group $D_\infty$. For a good match with formulas (28), we must take the renormalization $\psi = \sqrt{w} \cdot \Xi$ into account; then for the transformations $M_e$, $M_{e'}$, $M_{e''}$ this renormalization, i.e.,

$$\sqrt{w}\,\Xi \mapsto \sqrt{-w}\,\Xi = \mathrm{i}\sqrt{w}\,\Xi,$$

implies the transformation of matrices

$$\begin{pmatrix} 0 & -\beta \\ \beta^{-1} & 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & -(\mathrm{i}\beta) \\ (\mathrm{i}\beta)^{-1} & 0 \end{pmatrix} = -\mathrm{i} \begin{pmatrix} 0 & \beta \\ \beta^{-1} & 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & p \\ p^{-1} & 0 \end{pmatrix}.$$

We add a few words to the comments to example (36), (37). The fundamental difference between the finite-gap and not finite-gap case of equation (36) is expressed in their Galois groups. In the first case this group is solvable (in a specific way, and with transcendence degree 1), while in the second case, in the general position relative to the parameters $(\lambda, g_2, g_3)$, we deal with the general 3-parametric group $\mathrm{SL}_2(\mathbb{C})$.

[7] Strictly speaking, the Wronskian $\psi_z^+\psi^- - \psi^+\psi_z^- = 2\mu$ should be included in the differential relations as a field constant; the field itself must be hyperelliptically extended from $\mathbb{C}(\lambda)$ up to $\mathbb{C}(\lambda, \mu)$ (we disregard the other obvious constants).

**5.2. Algorithms and calculations.** As is well known, the Kovacic integration method branches into three mutually exclusive cases (the fourth corresponds to nonintegrability). The second (most "complicated") of these cases involves a quadratic extension of the field $\mathbb{C}(z)$, see [31, 35, 2]. This is precisely the case that corresponds to formula (14). Indeed, the Kovacic algorithm always leads to the algebraic irrationality

$$w = \sqrt{4z^3 - g_2 z - g_3}.$$

Thus, the finite-gap theory results in the fact that the algorithm, being applied to (11) for $Q_2 = 0$, is in fact a version of a more general situation which is not distinguished technically. There is also a formally theoretic extension of this algorithm to the fields of elliptic curves $\mathbb{C}(z, w)$ (Singer's algorithm [36, Theorem 4.3]). In the finite-gap case, these algorithms are efficiently realized on a very wide class of equations of the form (11), independently of the cases $Q_2 = 0$ or $Q_2 \neq 0$. The "unavoidable" elliptic integral (14) always occurs, so that, talking of monodromy and "complexity", it makes no sense to view the above two cases as distinct. For example, the property of the potential to be even may be lost or gained under the shift $x \mapsto x + \delta$ of the variable, and the Fuchs property on the plane/torus will be transferred to the torus/plane. Obviously, such transfers, changing monodromy representations, change nothing from the integrability viewpoint.

In other words, in our context the basic substitution $x \mapsto z = \wp(x)$ should be viewed as a special case of the more general substitution

$$z = \wp(\varepsilon x + \delta)$$

at least because the $x$-parameter itself on the torus is defined up to a linear transformation, and any finite-gap potential is only a representative of the equivalence class determined by the invariance of equation (1) and the corresponding (and even arbitrary) Novikov equation, see [17, 4, 23]. The last substitution is realized by the argument shift $x \mapsto x + \delta$. The scaling transformation $x \mapsto \varepsilon x$ does not change the form of the (finite-gap) equation (1), (13), in view of the renormalization $\lambda \mapsto \varepsilon^{-2}\lambda$, because $\lambda$ is arbitrary.

To complete our constructions, it remains to present a method for calculating the Hermite function[8] $\boldsymbol{R}(z, w; \lambda)$. For this, we use the recursive relations for the coefficients of the $\lambda$-series (7), which were deduced (in various forms) in the work of Gelfand and Dikiĭ (see, e.g., [4]). The simplest final formula (with all constants) was presented in [13] (there, like in [24], the quadrature nature of the finite-gap theory was indicated explicitly):

$$\boldsymbol{R} = \sum_{n=0}^{g} \lambda^n \sum_{j=0}^{g-n} c_j R_{g-n-j}, \quad R_0 = 1, \;\; c_0 = 1, \tag{41}$$

$$R_k = \frac{1}{8}\sum_{j=0}^{k-1} \{2(R_j)_{xx} R_{k-j-1} - (R_j)_x (R_{k-j-1})_x - 4u R_j R_{k-j-1}\} - \frac{1}{2}\sum_{j=1}^{k-1} R_j R_{k-j}. \tag{42}$$

It can be shown that this formula is equivalent to a spectral curve, i.e., to the fundamental equation (6) (this can also be seen from the structure of (41), (42)). Clearly, to adapt the situation to Fuchsian equations, now we should take the potential $u$ in any of the

[8]The algorithms themselves do not employ equation (6), which is the main object of the finite-gap theory; they are not tied to the "Fuchs" and "finite-gap" properties. They make use of the 3rd order equation (4), because the function $\boldsymbol{R}$ determines (see [31, p. 16] and [35, p. 134]) the second symmetric power of the operator $\partial_{zz} - U(z)$.

forms

$$u = Q_1(z) + wQ_2(z), \quad u = A_0 z + \sum_{\varrho\neq 0} A_\varrho \left\{ \wp'_\varrho \frac{\wp'_\varrho + w}{(z-\wp_\varrho)^2} + \frac{\wp''_\varrho}{z-\wp_\varrho} + 2\wp_\varrho \right\}. \tag{43}$$

**Theorem 9.** *For a given degree $g$ of the polynomial* (7)*, the "finite-gap" function $\boldsymbol{R} = \boldsymbol{R}(z, w; \lambda)$ of the general form* (41) *for the potential* (43) *is determined by the differential recurrence relation*

$$\begin{aligned} R_k = \frac{1}{8}\sum_{j=1}^{k-1} \Big\{ 2(w^2 R'_j)' R_{k-j-1} - (w^2 R_{k-j-1})' R'_j \\ - 4(R_{k-j} + uR_{k-j-1})R_j \Big\} - \frac{1}{2}uR_{k-1}, \end{aligned} \tag{44}$$

*where prime denotes the derivative $\frac{d}{dz}$, and where*

$$w^2 = 4(z-e)(z-e')(z-e''), \quad w_z = \frac{1}{2} w \ln_z (z-e)(z-e')(z-e'') .$$

*Proof.* In formula (42) we plug $\partial_x = w\frac{d}{dz}$ and $\partial_{xx} = w^2\frac{d^2}{dz^2} + ww_z\frac{d}{dz}$. Regrouping the terms yields (44). □

**Corollary 10.** *If $Q_2 = 0$, then $\boldsymbol{R}$ is a rational function in $z$ with possible poles only at the points $z = \{e, e', e'', \wp_\varrho, \infty\}$.*

*Remark* 8. A result similar Corollary 10 was presented in [39, Theorem 4], where normalization was chosen so that $\boldsymbol{R}$ be a polynomial in $z$. The recurrence relation (44) was written in a form ensuring that if $Q_2 = 0$, then the dependence $\boldsymbol{R}(z)$ is constructed automatically. If $Q_2 \neq 0$, then, in accordance with (14), a nontrivial contribution to monodromy (the elliptic integral) is given also by the "nonrational $\boldsymbol{R}_2$-part" of the function $\boldsymbol{R}$.

**Example 3.** The degree $g$ is equal to 1 or 2:

$$\begin{aligned} g = 1: \quad & \boldsymbol{R} = \lambda - \frac{1}{2}u + c_1, \\ g = 2: \quad & \boldsymbol{R} = \lambda^2 - \frac{1}{2}(u - 2c_1)\lambda - \frac{1}{8}(4z^3 - g_2 z - g_3)u'' \\ & \qquad - \frac{1}{16}(12z^2 - g_2)u' + \frac{3}{8}u^2 - \frac{1}{2}c_1 u + c_2. \end{aligned}$$

As a result, from an algorithmic point of view, we arrive to the following chain of actions. Let a potential $u$ (*Ansatz*) of the form (13) be given, along with a number $g$ ("genus"). We construct the recurrence relation (41), (44) and calculate the function $\boldsymbol{R}$. To find the constant $c_j$, we plug this $\boldsymbol{R}$ in (4), i.e., in the equation

$$(4z^3 - g_2 z - g_3)\boldsymbol{R}''' + \frac{3}{2}(12z^2 - g_2)\boldsymbol{R}'' + 4(3z - u - \lambda)\boldsymbol{R}' - 2u'\boldsymbol{R} = 0,$$

and convert the result into a polynomial in $z$. It must vanish identically. Equating its coefficients, which depend linearly on the $g$ constants $c_j$, we find these constants. If the equations are not compatible for any choice of the parameters $\{g_2, g_3, \varrho\}$, then the potential in question is not finite-gap. Next, the algebraic curve (6) and the constant $\mu$ are described by the formula

$$\mu^2 = \frac{1}{4}(4z^3 - g_2 z - g_3)\big\{(\boldsymbol{R}')^2 - 2\boldsymbol{R}\boldsymbol{R}''\big\} - \frac{1}{4}(12z^2 - g_2)\boldsymbol{R}\boldsymbol{R}' + (u + \lambda)\boldsymbol{R}^2,$$

where the right-hand side reduces automatically to the $\lambda$-polynomial (8) with the constant roots $E_j(g_2, g_3, \wp_\varrho, c_j)$. The solutions and monodromies are constructed by formulas (14) and (28), (29), in which the elliptic integrals are calculated as indicated in Theorem 1, by reducing to the forms (15). No hyperelliptic integrals arise.

Also, this argument implies automatically that if a Fuchsian equation is such that its principal part (in the presence of false singularities $z = \wp_\varrho$) and the accessory part coincide with those in (18) (possibly, after renormalization of the parameters), then this equation has a "finite-gap" origin. However, such an equation is integrated by the Kovacic algorithm, without any regard to the finite-gap theory.

Thus, *the finite-gap type potentials provide a natural way to increase the number of singularities* (*more than* 4) *in Fuchsian equations of the form* (11)$\Leftrightarrow$(18)*, with preservation of the fairly rigid property of integrability in Liouville extensions, and "leaving the parameter $\lambda$ arbitrary"*. The last requirement (we cited it from [24], where it was also marked out) can be viewed as a key one even for defining the finite-gap Fuchsian equations. Equations without parameters may come from arbitrarily complicated families of equations: Fuchsian, non-Fuchsian, integrable, nonintegrable, in any combinations. If a parameter is present, then the character of the dependence on it is substantial. For instance, the counterexample (36), (37) considered above, being equivalent to the problem $\Psi'' = \lambda\wp(x; 0, 1)\Psi$, gives rise to the equations

$$\psi_{zz} = \frac{\lambda z(4z^3 - 1) - 3z(z^3 + 2)}{(4z^3 - 1)^2}\psi\,, \quad \boldsymbol{\psi}_{ww} = \frac{1}{9}\frac{\lambda(w^2 + 1) - 2w^2 + 6}{(w^2 + 1)^2}\boldsymbol{\psi}.$$

They are not finite-gap Fuchsian and do not admit reduction to such equations by algebraic substitutions[9], though they are Fuchsian, integrable, and with computable monodromy. The first of them is a (normal) Heun equation, and the second is (normal) hypergeometric.

## §6. Concluding remarks

**Proposition 11.** *The finite-gap origin of a given Fuchsian equation can be established algorithmically.*

Indeed, if the equation is given over $\mathbb{C}(z)$, then, reshaping it to the normal form, we merely compare it with (11). Now, let some (normal) equation

$$Y_{uu} = P(u, v)Y, \tag{45}$$

on an elliptic curve be given, where the variables satisfy an algebraic relation $\Phi(u, v) = 0$ of genus $g = 1$. Suppose that this equation is Fuchsian, which can be detected easily by the form of the equation. We pass from $(u, v)$ to the pair $(\wp, \wp')$ using the parametrization $u = U(\wp, \wp')$, $v = V(\wp, \wp')$. This procedure is both ways algorithmic $(u, v) \rightleftarrows (\wp, \wp')$, i.e., the functions $\wp = S(u, v)$, $\wp' = T(u, v)$ are also computable. If the pair of invariants $(g_2, g_3)$ is reduced to the types $(1, 0)$, $(0, 1)$, or $(1, a)$, then the equation's form will be even unique. Now, consider the expression $S(u, v) = \wp$ as the algebraic change of variables $u \mapsto \wp$ in equation (45), viewing $\wp$ as a symbol. We construct a normal form equation for the function $\psi = \psi(\wp)$ determined via an analog of the scaling transformation (10):

$$Y = \left(U_\wp + \frac{12\wp^2 - g_2}{2\wp'}U_{\wp'}\right)^{\frac{1}{2}}\psi.$$

The equation will look like this:

$$\psi_{\wp\wp} = \big\{\boldsymbol{p}(\wp) + \wp'\boldsymbol{q}(\wp)\big\}\psi.$$

[9]More generally, by any substitutions that preserve the property to be Liouvillian.

Then, after the change $(\wp, \wp') \to (z, w)$, it suffices to check whether this equation is of type (11) and, if yes, integrate it.

If the equation in question has no parameter, but coincides with (11) structurally, then the parameter $\lambda$ can be inserted, thus embedding the equation in a finite-gap series. Another (equivalent) way consists in passing in (45) at once to the global parameter $x$ on the torus via the functions $\wp(x)$, $\wp'(x)$. Passing to the normal form and discarding the non-Fuchsian cases, we see that the only possible form of the equation looks like this:

$$\Psi_{xx} = \Big\{\sum\nolimits_{\varrho}\big(A_{\varrho}\wp(x-\varrho) + C_{\varrho}\zeta(x-\varrho)\big) + A_0\Big\}\Psi, \quad \sum\nolimits_{\varrho} C_{\varrho} = 0\,.$$

Obviously, this equation can be finite-gap only if all $C_{\varrho}$ are zero, $A_{\varrho} = n_{\varrho}(n_{\varrho}+1)$, and $A_0$ depends on $\lambda$ linearly. Choosing an appropriate scaling transformation $x \mapsto \varepsilon x + \delta$ and a pair of periods $2(\omega, \omega')$, we can reduce the final equation to the simplest form.

It is important to note the following: should we restrict the class under consideration to the rational type equations $Q_2 = 0$ only, we would have no possibility to establish integrability for the wide class of equations with $Q_2 \neq 0$, and even for the rational class coming from the preceding one via the shifts $x \mapsto x + \delta$.

## References


[1] N. I. Akhiezer, *Elements of the theory of elliptic functions*, 2nd ed., Nauka, Moscow, 1970. (Russian) MR0288319 (44:5517)

[2] L. M. Berkovich, *Factorization and transformations of differential equations*, R&C Dynamics, Moscow, 2002. (Russian)

[3] A. P. Veselov and A. B. Shabat, *A dressing chain and the spectral theory of the Schrödinger operator*, Funktsional. Anal. i Prilozhen. **27** (1993), no. 2, 1–21; English transl., Funct. Anal. Appl. **27** (1993), no. 2, 81–96. MR1251164 (94m:58179)

[4] I. M. Gel′fand and L. A. Dikiĭ, *Asymptotic properties of the resolvent of Sturm–Liouville equations, and the algebra of Korteweg–de Vries equations*, Uspekhi Mat. Nauk **30** (1975), no. 5, 67–100; English transl., Russian Math. Surveys **30** (1975), no. 5, 77–113. MR0508337 (58:22746)

[5] B. A. Dubrovin, *Theta-functions and nonlinear equations*, Uspekhi Mat. Nauk **36** (1981), no. 2, 11–80; English transl., Russian Math. Surveys **36** (1981), no. 2, 11–92 (1982). MR616797 (83i:35149)

[6] A. R. Its and V. B. Matveev, *Schrödinger operators with the finite-band spectrum and the* N*-soliton solutions of the Korteweg–de Vries equation*, Teoret. Mat. Fiz. **23** (1975), no. 1, 51–68; English transl., Theoret. and Math. Phys. **23** (1975), no. 1, 343–355 (1976). MR0479120 (57:18570)

[7] E. Kamke, *Differentialgleichungen. Lösungsmethoden und Lösungen.* I. *Gewöhnliche Differentialgleichungen*, Akad. Verlag, Zeipzig, 1959. MR0106302 (21:5036)

[8] I. Kaplansky, *An introduction to differential algebra*, Actualités Sci. Ind., No. 1251, Hermann, Paris, 1957. MR0093654 (20:177)

[9] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Ergeb. Math. Grenzgeb., Bd. 14, Springer-Verlag, New York–Heidelberg, 1972. MR0349820 (50:2313)

[10] H. Poincaré, *Sur les groupes des équations linéaires*, Acta Math. **4** (1884), no. 3, 201–312. MR1554639

[11] Yu. N. Sirota and A. O. Smirnov, *The Heun equation and the Darboux transformation*, Mat. Zametki **79** (2006), no. 2, 267–277; English transl., Math. Notes **79** (2006), no. 1–2, 244–253. MR2249115 (2008b:34005)

[12] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge Univ. Press, Cambridge, 1927. MR1424469 (97k:01072)

[13] N. V. Ustinov and Yu. V. Brezhnev, *On the* Ψ*-function for finite-gap potentials*, Uspekhi Mat. Nauk **57** (2002), no. 1, 167–168; English transl., Russian Math. Surveys **57** (2002), no. 1, 165–167. MR1914555 (2003b:34169)

[14] P. B. Acosta-Humánez, *Galoisian approach to supersymmetric quantum mechanics. The integrability analysis of the Schrödinger equation by means of differential Galois theory*, Verlag Dr. Muller, 2010.

[15] P. B. Acosta-Humánez, J.-J. Morales-Ruiz, and J.-A. Weil, *Galoisian approach to integrability of the Schrödinger equation*, Rep. Math. Phys. **67** (2011), no. 3, 305–374. MR2846216

[16] Acta Applicandae Mathematicae **36** (1994), 1–308 (all vol.).

[17] E. D. Belokolos, A. I. Bobenko, V. Z. Enol′skii, A. R. Its, and V. B. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, Springer-Verlag, 1994.

[18] F. Beukers and A. van der Waall A., *Lamé equations with algebraic solutions*, J. Differential Equations **197** (2004), no. 1, 1–25. MR2030146 (2004j:34202)

[19] F. Beukers, *Unitary monodromy of Lamé differential operators*, Regul. Chaotic Dyn. **12** (2007), no. 6, 630–641. MR2373163 (2008k:34341)

[20] Yu. V. Brezhnev, *What does integrability of finite-gap or soliton potentials mean*? Philos. Trans. R. Soc. Lond. Ser. A **366** (2008), no. 1867, 923–945. MR2377680 (2009i:37159)

[21] ———, *Spectral/quadrature duality*: *Picard–Vessiot theory and finite-gap potentials*, Algebraic Aspects of Darboux Transformations, Quantum Integrable Systems and Supersymmetric Quantum Mechanics, Contemp. Math., vol. 563, Amer. Math. Soc., Providence, RI, 2012, pp. 1–31. MR2905627

[22] G. Darboux, *Sur une équation linéaire*, Compt. Rend. Acad. Sci. **94** (1882), no. 25, 1645–1648.

[23] L. A. Dickey, *Soliton equations and Hamiltonian systems*, 2nd ed., Adv. Ser. Math. Phys., vol. 26, World Sci. Publ. Co., Inc., River Edge, NJ, 2003. MR1964513 (2004c:37160)

[24] J. Drach, *Détermination des cas de réduction de l'équation différentielle* $\frac{d^2y}{dx^2} = [\varphi(x)+h]y$, Compt. Rend. Acad. Sci. **168** (1919), 47–50.

[25] A. R. Forsyth, *Theory of differential equations*. Vol. 3, 4, Dover Publ., Inc., New York, 1959. MR0123757 (23:A1079)

[26] F. Gesztesy and R. Weikard, *Picard potentials and Hill's equation on a torus*, Acta Math. **176** (1996), no. 1, 73–107. MR1395670 (97f:14046)

[27] ———, *A characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy*, Acta Math. **181** (1998), no. 1, 63–108. MR1654775 (99k:14052)

[28] P. Ivanov, *On Lamé's equation of a particular kind*, J. Phys. A **34** (2001), 8145–8150. MR1871855

[29] F. Klein, *Ueber lineare Differentialgleichungen der zweiten Ordnung*, Vorlesung, gehalten im Sommersemester 1894, Göttingen, 1894 (manuscript).

[30] E. R. Kolchin, *Algebraic matric groups and the Picard–Vessiot theory of homogeneous linear ordinary differential equations*, Ann. of Math. (2) **49** (1948), 1–42. MR0024884 (9:561c)

[31] J. J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, J. Symbolic Comput. **2** (1986), no. 1, 3–43. MR839134 (88c:12011)

[32] R. S. Maier, *Algebraic solutions of the Lamé equation, revisited*, J. Differential Equations **198** (2004), no. 1, 16–34. MR2037748 (2004k:34181)

[33] ———, *Lamé polynomials, hyperelliptic reductions and Lamé band structure*, Philos. Trans. R. Soc. Lond. Ser. A **366** (2008), no. 1867, 1115–1153. MR2377687 (2009i:37175)

[34] J.-J. Morales-Ruiz, *Differential Galois theory and non-integrability of Hamiltonian systems*, Progr. Math., vol. 179, Birkhäuser Verlag, Basel, 1999. MR1713573 (2000g:37076)

[35] M. van der Put and M. F. Singer, *Galois theory of linear differential equations*, Grundlehren Math. Wiss., Bd. 328, Springer-Verlag, Berlin, 2003. MR1960772 (2004c:12010)

[36] M. F. Singer, *Liouvillian solutions of nth order homogeneous linear differential equations*, Amer. J. Math. **103** (1981), 661–682. MR623132 (82i:12028)

[37] M. F. Singer and F. Ulmer, *Liouvillian and algebraic solutions of second and third order linear differential equations*, J. Symbolic Comput. **16** (1993), no. 1, 37–73. MR1237349 (94i:34016)

[38] A. O. Smirnov, *Elliptic solitons and Heun's equation*, The Kowalevski Property (Leeds, 2000), CRM Proc. Lecture Notes, vol. 32, Amer. Math. Soc., Providence, RI, 2002, pp. 287–305. MR1916788 (2003f:34183)

[39] ———, *Finite-gap solutions of the Fuchsian equations*, Lett. Math. Phys. **76** (2006), 297–316. MR2238723 (2007g:34184)

[40] K. Takemura, *The Heun equation and the Calogero–Moser–Sutherland system.* III. *The finite-gap property and the monodromy*, J. Nonlinear Math. Phys. **11** (2004), no. 1, 21–46. MR2031210 (2005d:37124)

[41] L. W. Thomé, *Ueber lineare Differentialgleichungen mit mehrwerthigen algebraischen Coefficienten*, J. Reine Angew. Math. **115** (1895), 33–52, 119–149; **119** (1898), 131–147.

Tomsk State University, Lenin ave. 36, Tomsk 634050, Russia
*E-mail address*: brezhnev@mail.ru

Received April 5, 2011

Translated by A. PLOTKIN