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DYNAMIC INVESTMENT PORTFOLIO MODEL UNDER ASSET ALLOCATION CONSTRAINTS

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The asset management task under portfolio constraints is considered. We propose to use the model predictive control methodology in order to obtain feedback trading strategies under constraints. The numerical modelling results are presented.

Keywords: *investment portfolio, constraints, MPC, feedback trading strategies.*

Introduction

The investment portfolio (IP) management is an area of both theoretical interest and practical importance [1 – 6], so this task was investigated by many researches. Dynamic investment-consumption strategy that maximise some integrated utility function was investigated by Merton [1] and others [2]. Bielecki and Pliska [3] employed methods of risk-sensitive control theory to portfolio optimisation. Dombrovskii and others [4-8] investigated another control-theoretical approach, they proposed dynamic stochastic models of the IP and formulated the IP management problem as a tracking task for some reference portfolio with desired return.

It's known that realistic investment models consider constraints on the trading volume amounts. Taking into account the portfolio constraints in dynamic models we face with the curse of dimensionality. So we have to find an alternative to classical methods.

In the paper we consider the IP management task under portfolio constraints. IP evolution is described by the model proposed in [7]. We propose to use the model predictive control (MPC) methodology in order to solve the problem. The MPC proved to be an appropriate and effective technique to solve the dynamic control tasks under constraints [8-9]. We obtain feedback strategies of investment portfolio optimisation under trading volume constraints. We also present the numerical modelling results that give evidence of capacity and effectiveness of proposed approach.

1. The investment portfolio model

Let's consider an investment portfolio composed of n risky assets (stocks) and one risk-free asset (bank account, bonds). Total portfolio capital at moment k is equal to

$$V(k)=\sum_{i=1}^{n}u_i(k)+u_0(k), \qquad (1)$$

where $u_i(k)$ $(i=\overline{1,n})$ is i[th] stock holding, $u_0(k)$ is a capital invested in risk-free asset. Total portfolio capital at next moment k+1 is equal to

$$V(k+1)=\sum_{i=1}^{n}[1+\eta_i(k+1)]u_i(k)+[1+r(k+1)]u_0(k), \qquad (2)$$

where $r_1(k)$ is non-random time-varying yield of risk-free asset, $\eta_i(k)$ is stochastic return of the i[th] stock.

It should be noted that the case of $u_0(k)<0$ means the borrowing of risk-free capital in amount $|u_0(k)|$, and the case of $u_i(k)<0$ $\left(i=\overline{1,n}\right)$ means the short sale [1].

We also suppose that the following constraints should be satisfied:

$$u_i^{\min}(k)\le u_i(k)\le u_i^{\max}(k), \quad (i=\overline{0,n}). \qquad (3)$$

If $u_i^{\min}(k)\ge 0$ $(i=\overline{1,n})$ then short sale operations are prohibited; if $u_0^{\min}(k)\ge 0$ then borrowing of risk-free capital is impossible; we also can restrict amounts of investments by $u_i^{\max}(k)$ $(i=\overline{0,n})$.

It should be noted that $u_i^{\min}(k)$ and $u_i^{\max}(k)$ $\left(i=\overline{0,n}\right)$ may be both constant and function, for example $u_i^{\min}(k)=\gamma_i' V(k)$, $u_i^{\max}(k)=\gamma_i^{*} V(k)$, where γ_i', γ_i^{*} are constant coefficients.

Equation (1) can be written as

$$Cu(k)=V(k)-u_0(k), \qquad (4)$$

where C is the n-dimensional row vector whose entries are equal to 1.

From (4) it follows that [7, 10]:

$$u(k) = C^{+}(V(k) - u_0(k)) + (I_n - C^{+}C)\nu(k), \tag{5}$$

where I_n is n-dimensional identity matrix, $\nu(k) = [\nu_1(k), ..., \nu_n(k)]^T$ is a vector, C^+ is pseudoinverse matrix, $C^+ = C^T(CC^T)^{-1}$.

Using (5) write (2) as

$$V(k+1) = [1 + \bar{\eta}(k+1)]V(k) + b(k+1)\lambda(k), \tag{6}$$

where $b(k) = [r(k) - \bar{\eta}(k), \eta_1(k) - \bar{\eta}(k), ..., \eta_n(k) - \bar{\eta}(k)]$, $\lambda(k) = [u_0(k), \nu_1(k)..., \nu_n(k)]^T$ is assumed to be a control vector, $\bar{\eta}(k) = n^{-1}\sum_{i=1}^{n}\eta_i(k)$.

We use the following difference equation to describe risky financial asset price evolution

$$S_i(k+1) = S_i(k)\left[1 + \mu_i(k+1) + \sum_{j=1}^{n}\sigma_{ij}(k+1)\, w_j(k+1)\right], (i = \overline{1,n})$$

where $S_i(k)$ is i-th asset price, $\mu_i(k)$ is average return of the i-th stock (coefficient of growth), $w_i(k)$ are standard normal white noise shocks, $\sigma(k) = \|\sigma_{ij}(k)\|$ is matrix of volatilities. Hence the equation (6) becomes

$$V(k+1) = \left(1 + \bar{\mu}(k+1) + \sum_{j=1}^{n}\bar{\sigma}_j(k+1)w_j(k+1)\right)V(k) + \left(b_0(k+1) + \sum_{j=1}^{n}b_j(k+1)w_j(k+1)\right)\lambda(k),$$

where

$$b_0(k) = [r(k) - \bar{\mu}(k) \quad \mu_1(k) - \bar{\mu}(k) \quad ... \quad \mu_n(k) - \bar{\mu}(k)],$$

$$b_j(k) = [-\bar{\sigma}_j(k) \quad \sigma_{1j}(k) - \bar{\sigma}_j(k) \quad ... \quad \sigma_{nj}(k) - \bar{\sigma}_j(k)],$$

$$\bar{\sigma}_j(k) = n^{-1}\sum_{i=1}^{n}\sigma_{ij}(k), \ \bar{\mu}(k) = n^{-1}\sum_{i=1}^{n}\mu_i(k).$$

From (5) it's evident that

$$u_i(k) = \frac{1}{n}V(k) + \nu_i(k) - \frac{1}{n}\left(\sum_{l=1}^{n}\nu_l(k) + u_0(k)\right), \ (i = \overline{1,n}),$$

hence the constraints (3) become

$$u_0^{\min}(k) \le u_0(k) \le u_0^{\max}(k), \tag{7}$$

$$u_i^{\min}(k) \le \frac{1}{n}V(k) + \nu_i(k) - \frac{1}{n}\left(\sum_{j=1}^{n}\nu_j(k) + u_0(k)\right) \le u_i^{\max}(k), \ (i = \overline{1,n}). \tag{8}$$

2. Investment portfolio management strategies design

Model predictive control is an open-loop control design procedure where at each sampling time k, plant measurements are obtained and a model of the process is used to predict future states of the system. Using these predictions, p control moves $\lambda(k+i/k)$, $i = \overline{0, p-1}$ are computed by minimising a nominal cost over a prediction horizon p, subject to constraints on the state and control input. According to the receding principle, only $\lambda(k) = \lambda(k/k)$ is applied to the real plant, and the whole procedure starts again at the next sampling time $k+1$, with the new current process knowledge. As control input $\lambda(k/k)$ depends on current state of the system, so we obtain a feedback control law.

The investment portfolio management problem is formulated as a dynamic tracking task for some reference portfolio $V^0(k)$ [5 – 8] which evolution is described by

$$V^0(k+1) = \left[1 + \mu^0(k)\right]V^0(k), \tag{9}$$

where $\mu^0(k)$ is desired yield.

So at time k we consider the following quadratic objective [8]:

$$J = E\left\{\sum_{i=1}^{p}\left[V(k+i) - V^0(k+i)\right]^2 + \lambda^T(k+i-1/k)R(k+i-1)\lambda(k+i-1/k) \Big/ V(k)\right\}, \tag{10}$$

where $R(k) > 0$ is a symmetric positive definite matrix of control cost coefficients, p is a prediction horizon, $E\{../...\}$ denotes conditional expected value operator.

Theorem. Optimal predictive control vector $\Lambda(k/k)=\left[\lambda^T(k/k),\ \lambda^T(k+1/k),\ \dots\ \lambda^T(k+p-1/k)\right]^T$ minimising functional (10) subject to (6), (9) under constraints (7)-(8) is a decision of quadratic programming task with criterion

$$Y(\mathrm{k+p}/k)=\Lambda^T(k)H(k)\Lambda(k)+2x^T(k)G(k)\Lambda(k),$$

subject to constraints

$$\Lambda^{\min}(k)\le\overline{S}\Lambda(k)\le\Lambda^{\max}(k),$$

where $x(k)=\left[V(k)\ V^0(k)\right]^T$ and $\overline{S}$, $H(k)$, $G(k)$ are partitioned matrices

$$\overline{S}=\left[S\quad 0_{(n+1)\times(n+1)(p-1)}\right],$$

$$H(k)=\begin{bmatrix}H_{11}(k) & \dots & H_{1p}(k)\\ \dots & \dots & \dots\\ H_{p1}(k) & \dots & H_{pp}(k)\end{bmatrix},$$

$$G(k)=\left[G_1(k)\ \dots\ G_p(k)\right],$$

with submatrices:

$$S=I_{n+1}-\frac{1}{n}\begin{bmatrix}0_n & 0\\ C^TC & C^T\end{bmatrix},$$

0_n represents n-dimensional raw vector of 0's, $0_{(n+1)\times(n+1)(p-1)}$ is (n+1) by (n+1)(p-1) null matrix,

$$H_{is}(k)=\begin{cases}B_0^T(k+i-1)\left\{\prod_{j=i}^{s-2}A_0(k+j)\right\}\sum_{j=0}^{n}A_j(k+s-1)Q(p-s)B_j(k+s-1), & i<s,\\ \sum_{j=0}^{n}B_j^T(k+s-1)Q(p-s)B_j(k+s-1)+R(k+s-1), & i=s,\\ H_{si}^T, & i>s,\end{cases}$$

$$G_s(k)=\left\{\prod_{j=0}^{s-2}A_0(k+j)\right\}\sum_{j=0}^{n}A_j(k+s-1)Q(p-s)B_j(k+s-1),$$

$$\prod_{j=i}^{i-1}A_0(k+j)=I_{n+1},$$

$$Q(s+1)=\sum_{j=0}^{n}A_j(k+p-1-s)Q(s)A_j(k+p-1-s)+\overline{R},\ Q(0)=\overline{R},$$

$$\Lambda^{\min}(k)=\begin{bmatrix}u_0^{\min}(k)\\ u_1^{\min}(k)-n^{-1}V(k)\\ \dots\\ u_n^{\min}(k)-n^{-1}V(k)\end{bmatrix},\ \Lambda^{\max}(k)=\begin{bmatrix}u_0^{\max}(k)\\ u_1^{\max}(k)-n^{-1}V(k)\\ \dots\\ u_n^{\max}(k)-n^{-1}V(k)\end{bmatrix},$$

$$A_0(k)=diag\left(1+\overline{\mu}(k),1+\mu^0(k)\right),$$

$$A_j(k)=diag\left(\overline{\sigma}_j(k),0\right),$$

$$B_0(k)=\begin{bmatrix}b_0(k)\\ 0_n\end{bmatrix},$$

$$B_j(k)=\begin{bmatrix}b_j(k)\\ 0_n\end{bmatrix},$$

$$\overline{R}=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

Optimal control is

$$\lambda(k)=\left[I_{n+1},0_{(n+1)\times(n+1)(p-1)}\right]\Lambda(k).$$

3. Numerical modelling

Let's find the optimal structure of IP composed of bank account with yield $r_1 = 0.001$ and six kinds of securities with random yields $\eta_i(k) = \mu_i + \sigma_i w_i(k)$ $(i = \overline{1,6})$, where vector of average rates of return $\mu = [0.007, 0.0075, 0.008, 0.0085, 0.009, 0.010]$ matrix of volatilities $\sigma = diag(0.03,\ 0.035,\ 0.04,\ 0.045,\ 0.05,\ 0.06)$. The matrix of control cost coefficients is $R = 10^{-4} I_7$, reference portfolio yield $\mu_0 = 0.0085$. The short-selling and borrowing operations are prohibited, so $u_i^{\min}(k) = 0$ ($i = \overline{0,6}$). Predictive horizon p = 40, initial values $V(0) = V^0(0) = 1$.

Stocks prices are presented on fig. 1, where on abscissa axis time is indicated, while on axis of ordinates prices are indicated. Portfolio management results are presented on fig. 2, where on abscissa axis time is indicated, while on axis of ordinates investment sums are indicated. Here we can see that the capital of investment portfolio follows the capital of reference portfolio due to the dynamic distribution of the wealth between different kinds of assets.

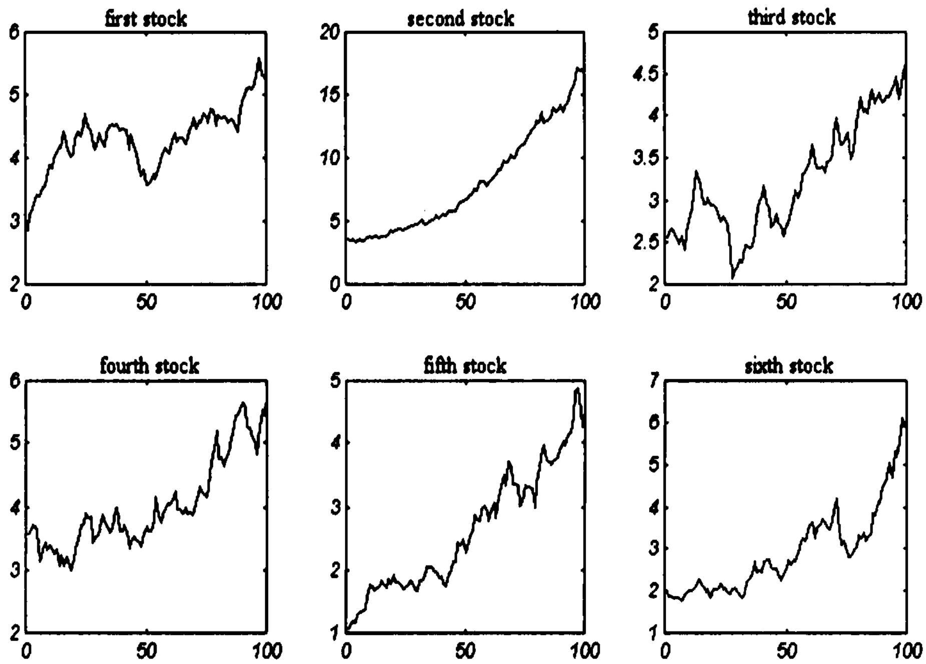

Fig. 1. Time series plots of the stocks prices

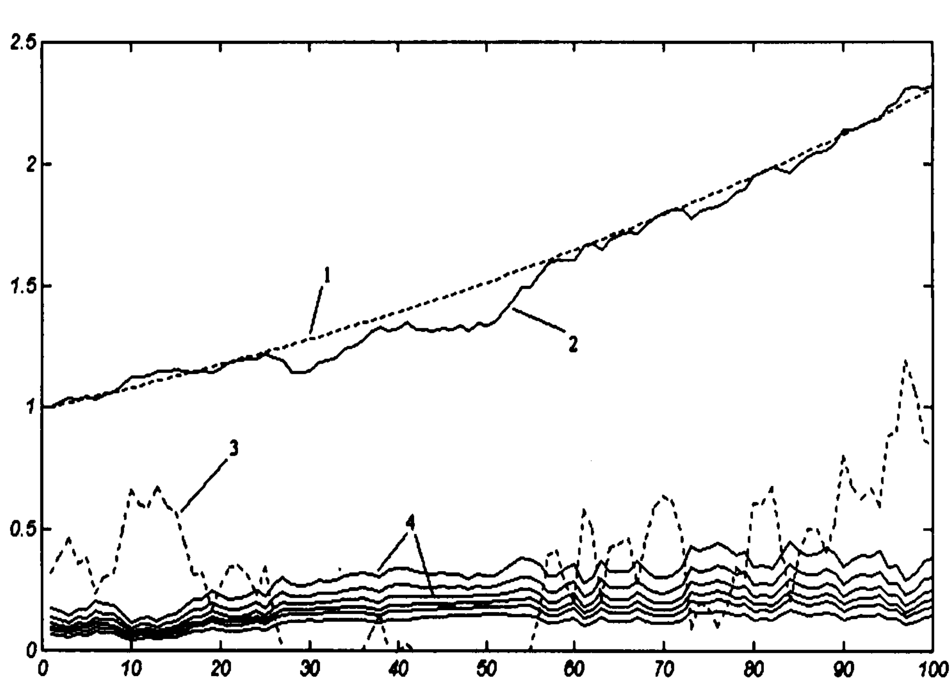

Fig. 2. Dynamic of portfolios capitals and amounts of investments: line 1 is reference portfolio capital, 2 is real portfolio capital, 3 is bank account evolution, lines 4 describe the evolution of the capital invested in the stocks

Conclusion

In the paper we consider the IP management task with transaction costs and portfolio constraints. We propose to use the model predictive control methodology in order to solve the problem. We obtain feedback strategies of investment portfolio optimisation with proportional transaction costs and trading volume constraints. Optimal trading strategies computation includes the decision of the sequence of quadratic programming tasks. The numerical modelling results give evidence of capacity and effectiveness of proposed approach.

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