ВЕСТНИК ТОМСКОГО ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА
2023 Математика и механика № 85
Tomsk State University Journal of Mathematics and Mechanics

Original article
UDC 519.22 MSC: 62G05, 62G20
doi: 10.17223/19988621/85/2

Super-efficient robust estimation in Lévy continuous time regression models from discrete data

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Аннотация. Abstract. In this paper we consider the nonparametric estimation problem for a continuous time regression model with non-Gaussian Lévy noise of small intensity. The estimation problem is studied under the condition that the observations are accessible only at discrete time moments. In this paper, based on the nonparametric estimation method, a new estimation procedure is constructed, for which it is shown that the rate of convergence, up to a certain logarithmic coefficient, is equal to the parametric one, i.e., super-efficient property is provided. Moreover, in this case, the Pinsker constant for the Sobolev ellipse with the geometrically increasing coefficients is calculated, which turns out to be the same as for the case of complete observations.
Keywords: nonparametric estimation, non-Gaussian regression models in continuous time, robust estimation, efficient estimation, Pinsker constant, super-efficient estimation

Acknowledgments: The research was carried out with the financial support of the RSF as part of a scientific project № 22-21-00302.

For citation: Nikiforov, N.I., Pergamenshchikov, S.M., Pchelintsev, E.A. (2023) Super-efficient robust estimation in Lévy continuous time regression models from discrete data. *Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika – Tomsk State University Journal of Mathematics and Mechanics*. 85. pp. 22–31. doi: 10.17223/19988621/85/2

Научная статья

Суперэффективное робастное оценивание в непрерывных регрессионных моделях Леви по дискретным данным

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Аннотация. Рассматривается задача непараметрического оценивания в модели непрерывной регрессии с негауссовским шумом Леви малой интенсивности. Задача оценивания изучается при условии, что наблюдения доступны только в дискретные моменты времени. На основе метода непараметрического оценивания строится новая процедура оценивания, для которой показано, что скорость сходимости до определенного логарифмического коэффициента равна параметрической, т.е. устанавливается свойство суперэффективности. Более того, в этом случае вычисляется константа Пинскера для соболевского класса с геометрически возрастающими коэффициентами, которая оказывается такой же, как и для случая полных наблюдений.
Ключевые слова: непараметрическое оценивание, модели негауссовской регрессии в непрерывном времени, робастное оценивание, эффективное оценивание, константа Пинскера, суперэффективное оценивание

Благодарности: Исследование выполнено при финансовой поддержке РНФ в рамках научного проекта № 22-21-00302.

Для цитирования: Никифоров Н.И., Пергаменщиков С.М., Пчелинцев Е.А. Суперэффективное робастное оценивание в непрерывных регрессионных моделях Леви по дискретным данным // Вестник Томского государственного университета. Математика и механика. 2023. № 85. С. 22–31. doi: 10.17223/19988621/85/2

1. Introduction

In this paper, we consider a non-Gaussian Lévy regression model in continuous time, introduced in [1], i.e.,

$$dy_t = S(t)dt + \varepsilon d\xi_t,\ 0 \le t \le 1, \tag{1.1}$$

where $S(\cdot)$ is a nonrandom unknown $[0,1] \to \mathbb{R}$ function from $\mathcal{L}_2[0,1]$, $(\xi_t)_{0\le t\le 1}$ is an unobserved noise defined through a Lévy process and $\varepsilon > 0$ is the noise intensity. We study the estimation problem for this model in nonparametric setting, i.e., we assume that

$$S(t) = \sum_{j=1}^{\infty} \theta_j \phi_j(t),\ 0 \le t \le 1, \tag{1.2}$$

where the Fourier coefficients $(\theta_j)_{j\ge1}$ belong to some set Θ defined later and $(\phi_j)_{j\ge1}$ is an orthonormal basis in $\mathcal{L}_2[0,1]$, i.e., for any $i, j \ge 1$

$$\left(\phi_i, \phi_j\right) = \int_0^1 \phi_i(t)\phi_j(t)dt = \mathbf{1}_{\{i\ne j\}}. \tag{1.3}$$

The problem is to develop efficient estimation methods for the regression function S, as $\varepsilon \to 0$, based on the discrete observations

$$(y_{t_l})_{0\le l\le n} \text{ and } t_l = \frac{l}{n}, \tag{1.4}$$

where the number of observations n is a function of the parameter ε, i.e., $n = n_\varepsilon$, such that $n_\varepsilon = O(\varepsilon^{-2})$ as $\varepsilon \to 0$. The condition $\varepsilon \to 0$ means that the signal/noise ratio goes to infinity. Note that, if $(\xi_t)_{0\le t\le 1}$ is a Brownian motion, then we obtain a "signal+white noise" model which is very popular in statistical radio-physics (see, for example, [2–4] and the references therein).

We assume that the stochastic component in the model (1.1) is given by a Lévy process with jumps. The reasons for the appearance of pulse noises in stochastic dynamic systems can be, for example, a sudden change in environmental conditions like the emergence of epidemics in sociological systems, crisis phenomena in economic systems, all kinds of failures and disruptions in the functioning of technical systems, etc. Note that the pulse noises for the continuous time regression models have been introduced in [5–7] on the basis of the compound Poisson processes for parametric regression models, and in [8, 9] such noises are used for nonparametric signal estimation problems. Later, to include all possible impulse noises, in the observation model (1.1) in [1] it is proposed to use general non-Gaussian Lévy processes $(\xi_t)_{0\le t\le 1}$ whose distribution Q is unknown and belongs to the distribution family Q^*_ε defined in the next section. For these reasons, in this paper, to study the quality of estimation, we use the robust risk

$$R^*_\varepsilon(\hat{S},S)=\sup_{Q\in Q^*_\varepsilon} R_Q(\hat{S},S)\,, \tag{1.5}$$

where $\hat{S}$ is an estimator (i.e., any measurable function of $(y_{t_l})_{0\le l\le n}$),

$$R_Q(\hat{S},S):=\mathbf{E}_Q\left\|\hat{S}-S\right\|^2 \text{ and } \|S\|^2=\int_0^1 S^2(t)dt\,. \tag{1.6}$$

Here $\mathbf{E}_Q$ stands the expectation with respect to distribution Q. We consider the minimax estimation problem, i.e., our main goal is to minimize the maximal value risk (1.5) over all possible estimation procedures $\hat{S}$, i.e.,

$$\sup_{S\in\Theta} R^*_\varepsilon(\hat{S},S)\to\min_{\hat{S}} \text{ as } \varepsilon\to 0\,.$$

To this end we use the exact lower bounds obtained in [10] for the nonparametric estimation problems on the basis of the complete data $(y_t)_{0\le t\le 1}$. It should be noted that in [10] the first time it was constructed super-efficient nonparametric estimation procedures, i.e., estimators for which the minimax convergence rate coincides with the parametric one up to a logarithmically increasing coefficient. In this paper, we show that the same lower bounds provide the super efficiency properties on the discrete observations (1.4) in the robust estimation setting.

The rest of the paper is organized as follows. In Section 2 we give the main conditions which will be assumed for the model (1.1). In Section 3 we construct the estimator. In Section 4 we state our main results on the adaptive efficiency. Section 5 contains the main proofs. Section 6 contains all necessary auxiliary results.

2. Main conditions

Let the unknown function S in (1.1) belong to the ellipse in $\mathcal{L}_2[0,1]$ defined as

$$\Theta=\left\{S\in\mathcal{L}_2[0,1]:\sum_{j=1}^{\infty}a_j\theta_j^2\le r\right\}, \tag{2.1}$$

where $a_j=e^{2\kappa j^\alpha}$ with fixed constants $0<\alpha<1$ and $\kappa>0$. For this set we need the following condition.

***A*₁)** $\forall S \in \Theta$ *there exists continuous derivate* $\dot{S}$ *such that* $\sup_{S\in\Theta}\|\dot{S}\| < \infty$.

To estimate unknown function S in (1.1) we use its Fourier expansion on the time grid $\{t_1,...,t_n\}$ defined in (1.4) for which we use empiric inner product and the norm in $\mathbb{R}^n$ defined as

$$\left(x,y\right)_n = \frac{1}{n}\sum_{j=1}^{n} x_j y_j \quad \text{and} \quad \|x\|_n^2 = (x,x)_n .$$

As to the basis in (1.2) we assume that the first n functions $(\phi_j)_{1\le j\le n}$ are orthonormal with respect to this product, i.e.,

$$(\phi_i,\phi_j)_n = \frac{1}{n}\sum_{l=1}^{n}\phi_i(t_l)\phi_j(t_l) = \mathbf{1}_{\{i=j\}} . \tag{2.2}$$

For example, one can take spline basis defined in [11] or the trigonometric basis $(Tr_j)_{j\ge1}$ with $Tr_1 \equiv 1$ and for $j \ge 2$

$$Tr_j(x) = \sqrt{2}\begin{cases}\cos(2\pi[j/2]x) & \text{for even } j,\\ \sin(2\pi[j/2]x) & \text{for odd } j,\end{cases} \tag{2.3}$$

where $[x]$ denotes the integer part of x. Note, that if n is odd, then the trigonometric basis possesses the orthonormality property (2.2). In this case we set $n = 2[\varepsilon^{-2}]+1$. Now, for any $t \in \{t_1,...,t_n\}$, we represent function S as

$$S(t) = \sum_{j=1}^{n}\theta_{j,n}\phi_j(t) \quad \text{and} \quad \theta_{j,n} = (S,\phi_j)_n . \tag{2.4}$$

***A*₂)** *For any* $\delta > 0$, *the coefficients* $(\theta_{j,n})_{1\le j\le n}$ *satisfy the following inequalities*

$$q_1 = \sup_{n\ge1}\max_{1\le j\le n} n \sup_{S\in\Theta}\frac{\left|\theta_{j,n}-\theta_j\right|}{j} < \infty \tag{2.5}$$

and

$$q_2 = \sup_{1\le N\le n}\sup_{S\in\Theta} n^2\left(\sum_{j=N}^{n}\theta_{j,n}^2 - (1+\delta)\sum_{j\ge N}\theta_j^2\right) < \infty . \tag{2.6}$$

Now we set

$$\varpi_{j,n} = \varpi_{j,n}(S) = \sum_{l=1}^{n}\int_{t_{l-1}}^{t_l}\phi_j(t_l)(S(u)-S(t_{l-1}))\,du . \tag{2.7}$$

***A*₃)** *The vector* $\left(\varpi_n\right)_{1\le j\le n}$ *is uniformly bounded in* $\mathbb{R}^n$, *i.e.*

$$q_3 = \sup_{n\ge1} n^2 \sup_{S\in\Theta}\sum_{j=1}^{n}\varpi_{j,n}^2 < \infty . \tag{2.8}$$

Remark 2.1. *Note that one can check directly that for the trigonometric basis* (2.3) *for all functions S from* Θ *and for any* $k \ge 1$ *there exists the continuous derivative of order k such that* $\sup_{S\in\Theta}\|S^{(k)}\|^2 < \infty$. *Therefore, Lemmas A.4–A.6 from* [9] *imply that the conditions* ***A*₂) – *A*₃)** *hold for the trigonometric basis.*

Furthermore, as to the noise process $(\xi_t)_{0\le t\le 1}$ similar to [1] we set

$$\xi_t = \rho_1 w_t + \rho_2 z_t \text{ and } z_t = x * (\mu - \tilde{\mu})_t , \tag{2.9}$$

where ρ_1 and ρ_2 are some unknown constants, $(w_t)_{0\le t\le 1}$ is a standard Brownian motion, "*" denotes the stochastic integral with respect to the compensated jump measure (see, for example, in [12], Chapter 3), $\mu(ds\,dx)$ is a jump measure with deterministic compensator $\tilde{\mu}(ds\,dx) - ds\Pi(dx)$, and $\Pi(\cdot)$ is the unknown Lévy measure such that

$$\Pi(x^2) = 1 \text{ and } \Pi(x^4) < \infty , \tag{2.10}$$

where $\Pi(|z|^m) = \int_{\mathbb{R}\setminus\{0\}} |z|^m \, \Pi(dz)$. Note that the measure $\Pi(\mathbb{R}\setminus\{0\})$ could be equal to $+\infty$. In the sequel we will denote by Q the distribution of the process $(\xi_t)_{0\le t\le 1}$ and by Q^*_ε the family of such distributions in the Skorokhod space $\mathbf{D}[0,1]$ for which

$$0 < \varsigma_* < \rho_1^2 \text{ and } \rho_1^2 + \rho_2^2 \le \varsigma^* \tag{2.11}$$

where the unknown bounds $0 < \varsigma_* \le \varsigma^*$ can be functions of ε, i.e., $\varsigma_* = \varsigma_*(\varepsilon)$ and $\varsigma^* = \varsigma^*(\varepsilon)$, such that for any $\delta > 0$

$$\liminf_{\varepsilon\to 0} |\ln \varepsilon|^{\delta} \varsigma_*(\varepsilon) > 0 \text{ and } \liminf_{\varepsilon\to 0} |\ln \varepsilon|^{-\delta} \varsigma^*(\varepsilon) < \infty . \tag{2.12}$$

In this case the expectation $\mathbf{E}_Q(\xi_t - \xi_s)^2 = (\rho_1^2 - \rho_2^2)(t-s)$ for any $0 < s < t < 1$ and, therefore, in view of the property (2.11)

$$\sup_{Q\in Q_\varepsilon} \mathbf{E}_Q(\xi_t - \xi_s)^2 \le \varsigma^*(t-s) . \tag{2.13}$$

The bounds $\varsigma_* \le \varsigma^*$ may be any positive fixed constants.

3. Estimation procedure

In this paper, as in [13], to estimate the function *S*, we use discrete Fourier expansion (2.4) in which we estimate the coefficients $(\theta_{k,n})_{1\le k\le n}$ through the least squares estimation method, i.e.,

$$\hat{\theta}_k = \sum_{j=1}^{n} \phi_k(t_j)\Delta y_{t_j} . \tag{3.1}$$

Using here the model (1.1), we obtain

$$\hat{\theta}_k = \theta_{k,n} + \varpi_{k,n} + \varepsilon\eta_k , \tag{3.2}$$

where $\eta_k = \sum_{j=1}^{n} \phi_k(t_j)\Delta\xi_{t_j}$. The orthonormality property (2.2) implies

$$\mathbf{E}_Q\eta_k^2 = \sum_{j=1}^{n} \phi_k(t_j)\mathbf{E}_Q\Delta\xi_{t_j}^2 ,$$

and, therefore, in view of the bounds (2.13)

$$\sup_{1\le k\le n} \sup_{Q\in Q_\varepsilon} \mathbf{E}_Q\eta_k^2 \le \varsigma^* . \tag{3.3}$$

Now, using the weighted least squares estimate from [10], we estimate the function S by

$$S^*_\varepsilon(t) = \sum_{k=1}^{n} \gamma_k \hat{\theta}_k \phi_k(t)\,,\ \gamma_k = (1 - e^{-\kappa(n^\alpha - k^\alpha)})\mathbf{1}_{\{1 \le k \le n_*\}} \tag{3.4}$$

and

$$n_* = \max\left\{1 \le l \le n : e^{2\kappa l^\alpha} g(l) \le \varepsilon^{-2} r\right\}, \tag{3.5}$$

where $g(l) = \sum_{j=1}^{l-1} e^{-\kappa(l^\alpha - j^\alpha)}\left(1 - e^{-\kappa(l^\alpha - j^\alpha)}\right)$.

4. Main results

First, we study the upper bound for the robust risk (1.5) corresponding to the estimation procedure (3.4).

Theorem 4.1. *Assume that the conditions* ***A*₁)–*A*₃)** *hold. Then*

$$\limsup_{\varepsilon \to 0} \upsilon_\varepsilon \sup_{S \in \Theta} R^*_\varepsilon(S^*_\varepsilon, S) \le \kappa^{-1/\alpha}, \tag{4.1}$$

where the rate $\upsilon_\varepsilon = \tilde{\varepsilon}^{-2} \left|\ln \tilde{\varepsilon}\right|^{-1/\alpha}$ *and* $\tilde{\varepsilon} = \varepsilon\sqrt{\varsigma^*}$.

Now, to compare with other estimators we need to introduce the class of possible estimators, i.e., let Ξ be the set of all estimators $\hat{S}$ measurable with respect to the σ-field generated by the observation (1.1), i.e., $\sigma\left\{y_{t_l}, 0 \le l \le n\right\}$.

Theorem 4.2. *The robust risk* (1.5) *admits the following lower bound*

$$\liminf_{\varepsilon \to 0} \upsilon_\varepsilon \inf_{\hat{S} \in \Xi} \sup_{S \in \Theta} R^*_\varepsilon(\hat{S}, S) \ge \kappa^{-1/\alpha}. \tag{4.2}$$

These theorems imply the following efficient property.

Theorem 4.3. *The estimate* (3.4) *is asymptotically efficient, i.e.,*

$$\lim_{\varepsilon \to 0} \frac{\inf_{\hat{S} \in \Xi_\varepsilon} \sup_{S \in \Theta} R^*_\varepsilon(\hat{S}, S)}{\sup_{S \in \Theta} R^*_\varepsilon(S^*_\varepsilon, S)} = 1 \tag{4.3}$$

and, moreover,

$$\lim_{\varepsilon \to 0} \upsilon_\varepsilon \sup_{S \in \Theta} R^*_\varepsilon(S^*_\varepsilon, S) = \kappa^{-1/\alpha}. \tag{4.4}$$

Remark 4.1. *For the model* (1.1) *the optimal convergence rate for parametric problems is* ε^2, *here we obtained* $\varepsilon^2 \left|\ln \varepsilon\right|^{1/\alpha}$, *i.e., almost parametric convergence rate up to the logarithmically increasing coefficient* $\left|\ln \varepsilon\right|^{1/\alpha}$. *The same effect was found in* [10] *for the case of continuous observations. For this reason, the estimation procedure* (3.4) *is called* ***super-efficient****.*

5. Proofs

5.1. **Proof of Theorem 4.1.** First note, that from (2.4), (3.2), and (3.3) one can deduce directly that

$$\mathbf{E}_Q \left\| S^*_\varepsilon - S \right\|^2_n = \varepsilon^2 \sum_{j=1}^{n} \gamma_j^2 \mathbf{E}_Q \eta_j^2 + U_n \ ,$$

where $U_n = \sum_{j=1}^{n} (\gamma_j \theta_{j,n} - \theta_{j,n} + \gamma_j \varpi_{j,n})^2$. Moreover, in view of the bound (3.3), we obtain

$$\sup_{Q \in Q_\varepsilon} \mathbf{E}_Q \left\| S^*_\varepsilon - S \right\|^2_n = \tilde{\varepsilon}^2 \sum_{j=1}^{n} \gamma_j^2 + U_n \ . \tag{5.1}$$

As to the last term, here note that for any $0 < \delta < 1$

$$U_n \le (1+\delta) T_n + (1+\delta^{-1}) \sum_{j=1}^{n} \varpi_{j,n}^2 \ , \tag{5.2}$$

where $T_n = \sum_{j=1}^{n} (1-\gamma_j)^2 \theta_{j,n}^2$ for which from (3.4) it follows that

$$T_n = \sum_{j=1}^{n_*} (1-\gamma_j)^2 \theta_{j,n}^2 + \sum_{j=n_*+1}^{n} \theta_{j,n}^2 := T_{1,n} + T_{2,n} \, .$$

Note here that for any $\delta > 0$

$$\theta_{j,n}^2 \le (1+\delta)\theta_j^2 + (1+\delta^{-1})(\theta_{j,n} - \theta_j)^2 \, .$$

In view of the condition (2.5), we obtain for any $0 < \delta < 1$

$$T_{1,n} \le (1+\delta) \sum_{j=1}^{n_*} (1-\gamma_j)^2 \theta_{j,n}^2 + q_1 (1+\delta^{-1}) n^{-2} \sum_{j=1}^{n_*} j^2 \le$$

$$\le (1+\delta) \sum_{j=1}^{n_*} (1-\gamma_j)^2 \theta_{j,n}^2 + q_1 (1+\delta^{-1}) n_*^3 n^{-2} \, .$$

Through the condition (2.6) we have

$$T_{2,n} \le (1+\delta) \sum_{j \ge n_*+1} \theta_j^2 + (1+\delta^{-1}) q_2 n^{-2}$$

and, we obtain

$$T_n \le (1+\delta) T_n^* + (1+\delta^{-1})(q_1 n_*^3 + q_2) n^{-2} \ ,$$

where the first term $T_n^* = \sum_{j \ge 1} (1-\gamma_j)^2 \theta_j^2$. To study the last term in (5.2), note that the condition (2.8) implies

$$\sup_{S \in \Theta} \sum_{j=1}^{n} \varpi_{j,n}^2 \le q_3 n^{-2} \, .$$

From Lemma 6.2

$$\limsup_{\varepsilon \to 0} \upsilon_\varepsilon \sup_{S \in \Theta} U_n \le \limsup_{\varepsilon \to 0} \upsilon_\varepsilon \sup_{S \in \Theta} T_n^* \, .$$

To estimate the term T_n^*, we apply the definition in (3.4). Then, it can be estimated as

$$T_n^* = \frac{1}{a_{n_*}} \sum_{j=1}^{n_*} a_j \theta_j^2 + \sum_{j \ge n_*+1} \theta_j^2 \le \frac{1}{a_{n_*}} \sum_{j \ge n_*} a_j \theta_j^2 \le \frac{r}{a_{n_*}} \, .$$

From the definition of n_* in (3.5) it follows that

$$a_{n_*+1} \ge r \varepsilon^{-2} / g(n_*+1) \ \text{ and } \ T_n^* \le \frac{a_{n_*+1}}{a_{n_*}} \varepsilon^2 g(n_*+1) \, .$$

Taking into account here that $\lim_{\varepsilon\to 0} a_{n_*+1}/a_{n_*}=1$ and using the last property in (6.2), we obtain that for any $0<\nu<\alpha$

$$\limsup_{\varepsilon\to 0}\frac{T_n^*}{\varepsilon^2 n_*^{1-\nu}}<\infty .$$

Therefore, the first property in (2.12) and Lemma 6.2 imply

$$\lim_{\varepsilon\to 0}\nu_\varepsilon T_n^*=0 .$$

Using the property (6.3) and Lemma 6.2 in the upper bound (5.1), we obtain that

$$\limsup_{\varepsilon\to 0}\upsilon_\varepsilon \sup_{S\in\Theta}\sup_{Q\in Q_\varepsilon}\mathbf{E}_Q\left\|S_\varepsilon^*-S\right\|_n^2\le\kappa^{-1/\alpha} .$$

Now Condition $\boldsymbol{A_1}$**)** and Lemma 6.3 imply the upper bound (4.1). □

5.2. **Proof of Theorem 4.2.** First of all, note that

$$R_\varepsilon^*(\hat{S},S)\ge\mathbf{E}_{Q_0}\left\|S_\varepsilon^*-S\right\|^2 ,$$

where Q_0 is the distribution of the noise $(\xi_t)_{0\le t\le 1}$ in (1.1) with $\rho_1=\sqrt{\varsigma^*}$ and $\rho_2=0$, i.e., under the distribution Q_0 we obtain the “signal+white noise” model, i.e., $dy_t=S(t)dt+\tilde{\varepsilon}dw_t$ with the small parameter $\tilde{\varepsilon}=\varepsilon\sqrt{\varsigma^*}$. So, Theorem 1 and Theorem 5 from [10] imply immediately the lower bound (4.2). □

6. Auxiliary results

Lemma 6.1. *The function* $g(n)$ *defined in* (3.5) *satisfies the following properties*

$$\liminf_{n\to\infty} n^{-1+\alpha}g(n)>0 , \tag{6.1}$$

for any $0<\nu<\alpha$

$$\limsup_{n\to\infty} n^{-1+\nu}g(n)<\infty . \tag{6.2}$$

Moreover, for the weight (3.4)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\gamma_j^2=1 . \tag{6.3}$$

Proof. Setting $\lambda_j=n^\alpha-(n-j)^\alpha$ we can represent g as

$$g(n)=\sum_{j=1}^{n-1}e^{-\kappa\lambda_j}\left(1-e^{-\kappa\lambda_j}\right)\ge\sum_{j=1}^{m}e^{-\kappa\lambda_j}\left(1-e^{-\kappa\lambda_j}\right),$$

where $m=[n^{1-\alpha}]$. For $n\ge 2^{1/\alpha}$ and $1\le j\le m$ through the Taylor expansion one can obtain that

$$\alpha\frac{j}{n^{1-\alpha}}\le\lambda_j\le\alpha\frac{j}{n^{1-\alpha}}+\frac{j^2}{n^{2-\alpha}}\le\alpha\frac{j}{n^{1-\alpha}}+1 .$$

Moreover, taking into account that $m\le n^{1-\alpha}\le 2m$, we obtain that

$$g(n)\ge e^{-\kappa}\sum_{j=1}^{m}e^{-\kappa\alpha\frac{j}{m}}\left(1-e^{-\frac{\kappa\alpha}{2}\frac{j}{m}}\right)$$

and, therefore,

$$\liminf_{n\to\infty} \frac{g(n)}{m} \geq e^{-\kappa} \int_0^1 e^{-\kappa\alpha u} \left(1 - e^{-\frac{\kappa\alpha}{2}u}\right) du > 0 .$$

This implies the lower bound (6.1). Moreover, note also that for any $0 < \rho < 1$

$$g(n) \leq (1-\rho)n + 1 + \sum_{j=1}^{[\rho n]} e^{-\kappa(n^{\alpha} - j^{\alpha})} \leq (1-\rho)n + 1 + ne^{-\kappa(1-\rho^{\alpha})n^{\alpha}} .$$

Choosing $\rho = 1 - n^{-\nu}$ with $0 < \nu < \alpha$ and noting through Taylor expansion that $1 - \rho^{\alpha} \geq \alpha n^{-\nu}$, as $n \to \infty$, we obtain the upper bound (6.2). Moreover, from this through the definition of weights γ_j immediately follows the property (6.3). □

Lemma 6.2. *The function* n_* *defined in* (3.4) *satisfies the following limit property*

$$\lim_{\varepsilon\to\infty} \frac{n_*}{|\ln\tilde{\varepsilon}|^{1/\alpha}} = \kappa^{-1/\alpha} .$$

Proof. First note that $n_* \to \infty$ as $\varepsilon \to 0$. Moreover, from (3.5) and (6.1) it follows that for a sufficiently small ε

$$2\kappa n_*^{\alpha} \leq \ln \varepsilon^{-2} r \text{ and } 2\kappa(n_* + 1)^{\alpha} + \ln g(n_* + 1) > \ln \varepsilon^{-2} r .$$

From here we can deduce immediately that $n_* / |\ln\varepsilon|^{1/\alpha} \to \kappa^{1/\alpha}$ as $\varepsilon \to 0$. Using here the bounds (2.12), we obtain this lemma. □

Lemma 6.3. *Let* $f : [0,1] \to \mathbb{R}$ *be an absolutely continuous function with square integrable derivative* $\dot{f}$ *, i.e.,* $\|\dot{f}\| < \infty$ *and* $g : [0,1] \to \mathbb{R}$ *be a piecewise constant function of the form* $g(t) = \sum_{j=1}^{n} c_j \chi_{(t_{j-1},\theta_j)}(t)$ *, where* c_j *are some constants. Then, for any* $\delta > 0$ *the function* $\Delta = f - g$ *satisfies the following inequality*

$$\|\Delta\|_n^2 \leq (1+\delta)\|\Delta\|^2 + (1+\delta^{-1}) \frac{\|\dot{f}\|^2}{n^2} .$$

References

1. Beltaief S., Chernoyarov O.V., Pergamenshchikov S.M. (2020) Model selection for the robust efficient signal processing observed with small Levy noise. *Annals of the Institute of Statistical Mathematics*. 72. pp. 1205–1235.
2. Ibragimov I.A., Khasminskii R.Z. (1981) *Statistical Estimation: Asymptotic Theory*. New York: Springer.
3. Kutoyants Yu.A. (1994) *Identification of Dynamical Systems with Small Noise*. Dordrecht: Kluwer Academic Publishers.
4. Pinsker M.S. (1981) Optimal filtration of square integrable signals in Gaussian white noise. *Problems of Transmission Information*. 17. pp. 120–133.
5. Kassam S.A. (1988) *Signal Detection in Non-Gaussian Noise*. New York: Springer-Verlag.
6. Konev V., Pergamenshchikov S., Pchelintsev E. (2014) Estimation of a regression with the impulse type noise from discrete data. *Theory of Probability and its Applications*. 58(3). pp. 442–457.

7. Pchelintsev E. (2013) Improved estimation in a non-Gaussian parametric regression. *Statistical Inference for Stochastic Processes*. 16(1). pp. 15–28.
8. Konev V.V., Pergamenshchikov S.M. (2012) Efficient robust nonparametric estimation in a semimartingale regression model. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*. 48(4). pp. 1217–1244.
9. Konev V.V., Pergamenshchikov S.M. (2015) Robust model selection for a semimartingale continuous time regression from discrete data. *Stochastic Processes and their Applications*. 125. pp. 294–326.
10. Pchelintsev E.A., Pergamenshchikov S.M., Povzun M.A. (2022) Efficient estimation methods for non-Gaussian regression models in continuous time. *Annals of the Institute of Statistical Mathematics*. 74. pp. 113–142.
11. Demmler A., Reinsch C. (1975) Oscillation matrices with spline smoothing. *Numerische Mathematik*. 24. pp. 357–382.
12. Liptser R., Shiryayev A.N. (1989) *Theory of Martingales.* Dordrecht: Kluwer Academic Publishers.
13. Pchelintsev E.A., Pergamenshchikov S.M., Leshchinskaya M.A. (2022) Improved estimation method for high dimension semimartingale regression models based on discrete data. *Statistical Inference for Stochastic Processes*. 25(3). pp. 537–576.

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The article was submitted 05.08.2023; accepted for publication 10.10.2023

Статья поступила в редакцию 05.08.2023; принята к публикации 10.10.2023